ZZU102 ENGINEERING MECHANICS II—DYNAMICS

MODULE 4

Kinetics of Plane Motion of Rigid Bodies

Introduction

We saw earlier in kinematics that the motion of a rigid body can be considered as the superposition of a translational and rotational motion at any time instant. The axis of rotation then passes through this chosen reference point. A convenient point to choose is the centre of mass.

For the translational motion, we may use the concepts of particle dynamics. Then, we have

$$\mathbf{F} = M\,\dot{\mathbf{V}}_c\,, \tag{4.1}$$

where M is the total mass of the body and $\mathbf{V}_c$ is the velocity of the centre of mass. For the rotary motion, we have

$$\mathbf{M}_A = \dot{\mathbf{H}}_A\,. \tag{4.2}$$

The point A in the above can be the mass centre, a point fixed in an inertial reference, or a point accelerating toward or away from the mass centre. For these points, the angular velocity $\boldsymbol{\omega}$ is also involved. Moreover, the inertia tensor is involved.

Moment of Momentum Equations—General Rigid Body Motion

Consider a rigid body moving arbitrarily relative to an inertial reference XYZ as shown in Fig. 4.1. Choose any point A within the body or on a hypothetical massless rigid body extension of the body. Consider an infinitesimal element of mass dm at position $\boldsymbol{\rho}$ in the body as shown in the figure. The velocity $\mathbf{V}'$ of the elemental mass dm relative to A is simply the velocity of dm relative to any reference $\xi\eta\zeta$ which translates with A relative to XYZ.

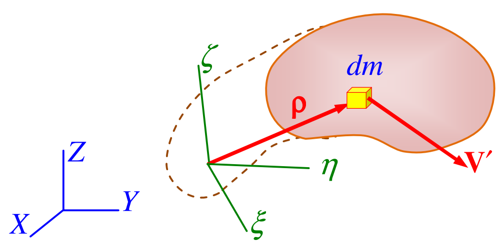

Figure 4.1

The linear momentum of dm relative to A (i.e. $\mathbf{V}'dm$) is the linear momentum of dm relative to $\xi\eta\zeta$ translating with A. The moment of this momentum (i.e. the angular momentum) $d\mathbf{H}_A$ about A can be obtained as

$$d\mathbf{H}_A = \boldsymbol{\rho}\times\mathbf{V}'dm = \boldsymbol{\rho}\times\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{\xi\eta\zeta} dm\,.$$

Now, since A is fixed in the body (or on a hypothetical extension of the body), and dm is a part of the body, the vector $\boldsymbol{\rho}$ is fixed in the body. Hence

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{\xi\eta\zeta} = \boldsymbol{\omega}\times\boldsymbol{\rho}\,,$$

where $\boldsymbol{\omega}$ is the angular velocity of the body relative to $\xi\eta\zeta$. But, as $\xi\eta\zeta$ translates with respect to XYZ, $\boldsymbol{\omega}$ is the angular velocity of the body with respect to XYZ as well. Hence

$$d\mathbf{H}_A = \boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})dm .$$

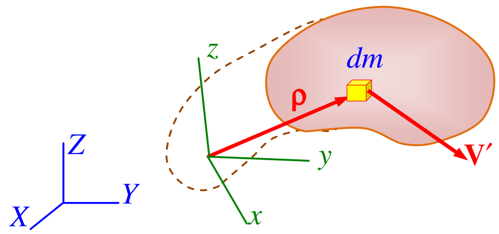

Figure 4.2

Now, we ignore $\xi\eta\zeta$ and use xyz (fixed to the body at A) instead. Thus, let us now consider Fig. 4.2. Integrating the above equation over the mass of the body yields

$$\mathbf{H}_A = \int_m \boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})dm = [I_x]\boldsymbol{\omega} , \tag{4.3a}$$

where $[I_x]$ is the inertia tensor. The above in expanded form appears as

$$\begin{Bmatrix} H_{Ax} \\ H_{Ay} \\ H_{Az} \end{Bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} . \tag{4.3b}$$

In the above, the 3×3 matrix represents the inertia tensor with two sets of components. The diagonal elements being the mass moments of inertia as given by

$$I_{xx} = \int_M (y^2+z^2)dm ,\ I_{yy} = \int_M (x^2+z^2)dm \text{ and } I_{zz} = \int_M (x^2+y^2)dm , \tag{4.4a}$$

and the off-diagonal elements corresponding to the mass products of inertia given by

$$I_{xy} = \int_M xy\,dm ,\ I_{yz} = \int_M yz\,dm \text{ and } I_{xz} = \int_M xz\,dm . \tag{4.4b}$$

In the case of general motion of the rigid body, the above turns out to be: $\mathbf{H}_A = [I_x]\ \boldsymbol{\omega}$, where $[I_x]$ is the mass-inertia tensor.

Now, we need to employ the moment of momentum equation, viz. Eq. (4.2). Recall that the point A can be,

1) the moving centre of mass of the body,
2) a '*fixed*' point or a point which is moving with a constant velocity at " t " in XYZ, or
3) a point that is accelerating toward or away from the mass centre at t.

Now, we can write

$$\left(\frac{d\mathbf{H}_A}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{H}_A}{dt}\right)_{xyz} + \boldsymbol{\omega}\times\mathbf{H}_A , \tag{4.5}$$

where $\boldsymbol{\omega}$ is the angular velocity of xyz (and thus of the body) relative to XYZ. Hence, we have

$$\mathbf{M}_A = \left(\frac{d\mathbf{H}_A}{dt}\right)_{xyz} + \boldsymbol{\omega}\times\mathbf{H}_A , \tag{4.6a}$$

or

$$\mathbf{M}_A = [I_x]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\left([I_x]\boldsymbol{\omega}\right). \tag{4.6b}$$

Plane Motion of Rigid Bodies

In the case of planar motion, we have $\boldsymbol{\omega} = \omega\mathbf{k}$, where $\mathbf{k}$ is the unit vector along x-axis. As a result, Eq. (4.3) yields

$$\mathbf{H}_A = \begin{Bmatrix} H_{Ax} \\ H_{Ay} \\ H_{Az} \end{Bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} = \begin{Bmatrix} -I_{xz} \\ -I_{yz} \\ I_{zz} \end{Bmatrix}\omega. \tag{4.7}$$

Similarly, Eq. (4.6) for this case leads to

$$\mathbf{M}_A = [I_x]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\left([I_x]\boldsymbol{\omega}\right) = \begin{Bmatrix} -I_{xz} \\ -I_{yz} \\ I_{zz} \end{Bmatrix}\dot{\omega} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ -\omega I_{xz} & -\omega I_{yz} & \omega I_{zz} \end{bmatrix},$$

which can be written as

$$\mathbf{M}_A = \begin{Bmatrix} -I_{xz} \\ -I_{yz} \\ I_{zz} \end{Bmatrix}\dot{\omega} + \begin{Bmatrix} I_{yz} \\ I_{xz} \\ 0 \end{Bmatrix}\omega^2. \tag{4.8}$$

Note:

- The angular velocity vector $\boldsymbol{\omega}$ is always taken relative to the inertial reference *XYZ*.
- On the other hand, the moment of forces (i.e. $\mathbf{M}_A$), is taken about *xyz* fixed to the body at *A*.
- For all plane motions, the last equation of (4.8), viz. $(M_A)_Z = I_{ZZ}\dot{\omega}$, is always the same; the other two may get modified.

Pure Rotation of a Body of Revolution about its Axis of Revolution

Consider a uniform body of revolution as shown in Fig. 4.3. If it undergoes pure rotation about the axis of revolution fixed in an inertial space *XYZ*, we have plane motion parallel to any plane normal to the axis of revolution. The *xyz*-system is fixed such that *z*-axis is collinear with the axis of revolution. The *xyz* axes can have any arbitrary orientation with respect to *XYZ*; however, let us choose them to be collinear with *XYZ* at *t*.

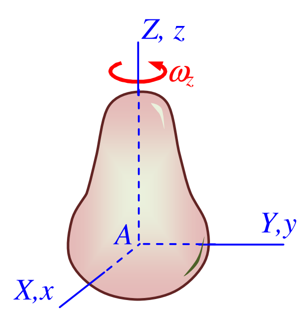

Figure 4.3

As the body is a solid of revolution, we have $I_{xy} = I_{yz} = I_{xz} = 0$. Hence, the *xyz*-axes are the principal axes. As a result, from Eq. (4.8), we have

$$M_Z = I_{ZZ}\,\dot{\omega}_Z \text{ and } M_x = M_y = 0. \tag{4.9}$$

As centre of mass is on the axis of revolution, it is stationary. Hence we have

$$\sum F_x = 0,\ \sum F_y = 0 \text{ and } \sum F_z = 0. \tag{4.10}$$

From Eq. (4.9), we have

$$M_Z = I_{ZZ}\ddot{\theta}, \tag{4.11}$$

which is similar to the Newton's law statement $\mathbf{F} = m\ddot{x}$.

Ex: 4.1 A stepped cylinder as shown in Fig. 4.4 has the dimensions R_1 = 0.3 m, R_2 = 0.65 m, and the radius of gyration, k = 0.35 m. The mass of the stepped cylinder is 100 kg. Blocks A and B are connected to the cylinder. If block B is of mass 80 kg and block A is of mass 50 kg, how far does A move in 5 sec? In which direction does it move?

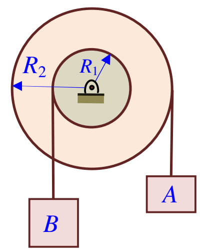

Figure 4.4

Let $\ddot{\theta}$ be the angular acceleration of the cylinder in the counter-clockwise direction as shown in the free body diagrams given in Fig. 4.5. The D'Alembert's fictitious inertia forces are also shown in all the three free body diagrams.

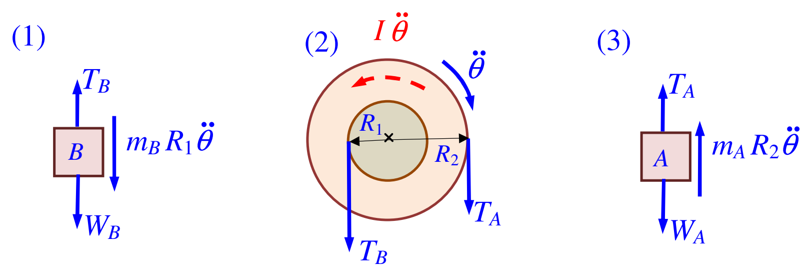

Figure 4.5

From free body diagram (1), we have

$$T_B = W_B + m_B R_1 \ddot{\theta};$$

from the free body diagram (3), we obtain

$$T_A = W_A - m_A R_2 \ddot{\theta};$$

and from free body diagram (2), we get

$$T_A R_2 - T_B R_1 = I\ddot{\theta}.$$

Using the first two equations in the third yields

$$m_A (g - R_2\ddot{\theta}) R_2 - m_B (g + R_1\ddot{\theta}) R_1 = I\ddot{\theta}.$$

Therefore, we get

$$\ddot{\theta}\left\{I + m_A R_2^2 + m_B R_1^2\right\} = m_A g R_2 - m_B g R_1.$$

Substituting the values, we obtain

$$\ddot{\theta}\left\{100\times 0.35^2 + 50\times 0.65^2 + 80\times 0.3^2\right\} = 50\times 9.81\times 0.65 - 80\times 9.81\times 0.3,$$

which leads to the solution, $\ddot{\theta}$ = 2.0551 rad/s. Thus, the assumed direction of rotation is correct, and therefore body A descends.

Now, $a_A = R_2\ddot{\theta} = 1.336\,\text{m/s}^2$. From this, we can get the velocity as $V_A = 1.336t + A$. As $V_A(t = 0) = 0$, $A = 0$. Integrating once again, we get the position as $y_A = 1.336t^2/2 + B$, where $B = 0$. Therefore, $y_A\ (t = 1.5\text{ s}) = 16.6975$ m.

Pure Rotation of a Body with Two Orthogonal Planes of Symmetry

Consider a uniform body with two orthogonal planes of symmetry as depicted in Fig. 4.6.

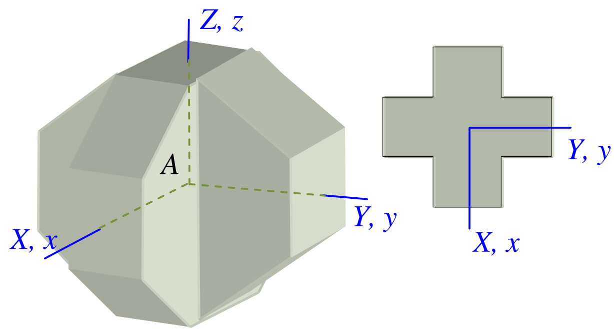

Figure 4.6

Consider its pure rotation about an axis that is stationary and collinear with the intersection of the planes of symmetry. Let us denote it by the z-axis. The origin A can be fixed anywhere on the z-axis; x and y-axes are chosen in the plane of symmetry. The XYZ-axes are chosen collinear with xyz at time t. The equations of motion work out to be identical to the previous case.

Pure Rotation of Slab-Like Bodies

Consider a slab-like body having a single plane of symmetry as shown in Fig. 4.7. The plane of symmetry is along the xy-plane. Consider pure rotation of the body about an axis normal to the xy-plane. As the z-axis is normal to the xy-plane which is a plane of symmetry, we have $I_{xz} = I_{yz}$ 0.

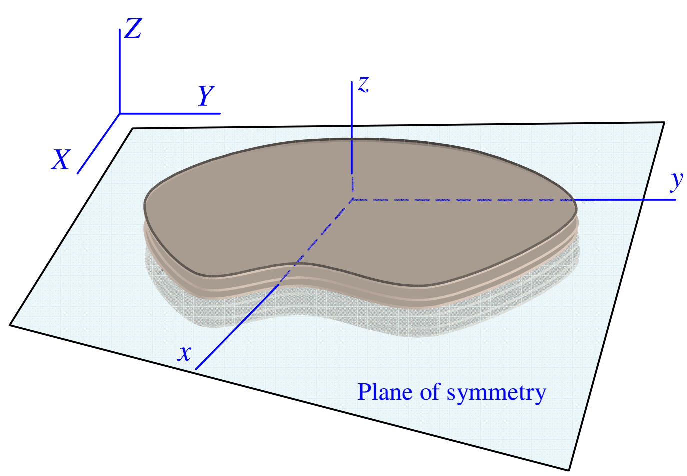

Figure 4.7

Thus, the moment of momentum equations are

$$M_x = 0,\ M_y = 0 \text{ and } M_z = I_{zz}\,\dot{\omega}_z = I_{zz}\ddot{\theta}. \tag{4.12}$$

If the centre of mass is not along the axis of rotation, we do not have equilibrium for the centre of mass; it undergoes circular motion. We need to use Newton's laws and kinematics relations to solve this problem. If the axes *xyz* are the principal axes for the body, we again obtain the same set of equations as before provided *z*-axis is the axis of rotation.

Ex: 4.2 A plate weighing 3 lb/ft^2 is supported at *A* and *B* as shown in Fig. 4.8a. What are the force components at *B* at the instant support *A* is removed?

Consider the free body diagram shown in Fig. 4.8b. We have, tan $\theta = 0.8$, and

$$I_{zz} = \rho t\left(\frac{bd^3}{3} + \frac{db^3}{3}\right) = \frac{\rho bdt}{3}\left(d^2 + b^2\right) = \frac{8\times 10\times 3}{3g}\left(10^2 + 8^2\right) = 407.453.$$

Now, $\sum M_z = 0$:

$$I_{zz}\ddot{\theta} = W\times 4 = 3\times 80\times 4. \text{ Therefore, } \ddot{\theta} = 2.3561 \text{ rad/s}^2.$$

Let the centre of mass have the accelerations $\ddot{X}$ and $\ddot{Y}$. Thence, we have

$$|\mathbf{a}| = \left|\ddot{X}\,\mathbf{i} + \ddot{Y}\,\mathbf{j}\right| = \sqrt{4^2 + 5^2}\,\ddot{\theta} = 6.40312\ddot{\theta} = 15.0864 \text{ ft/s}^2.$$

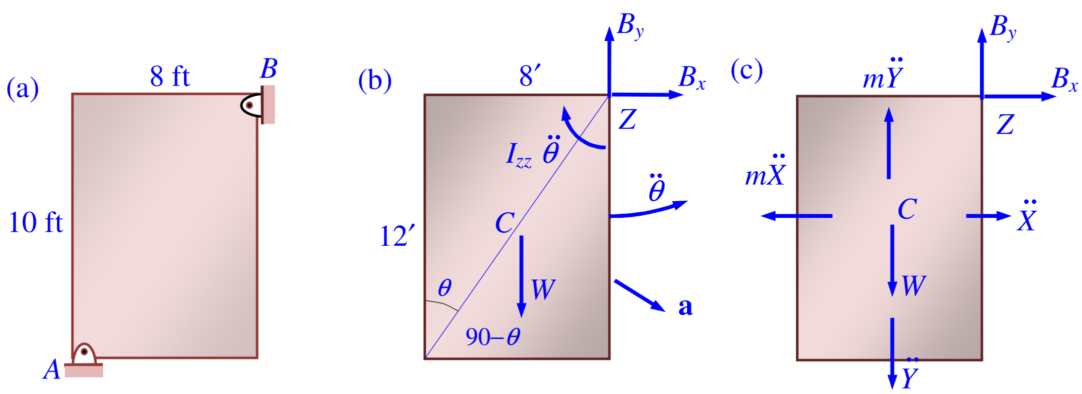

Figure 4.8

Therefore, $\ddot{X} = a\cos\theta = 11.7805$ m/s and $\ddot{Y} = a\sin\theta = 9.4244$ m/s.

Now considering the second free body diagram shown as Fig. 4.8c, we obtain

$$\sum F_x = 0 : B_x = m\ddot{x} = \underline{37.805} \text{ lb.}$$

and

$$\sum F_y = 0 : B_y = W - m\ddot{y} = \underline{169.756} \text{ lb.}$$

Rolling Slab-like Bodies

Consider rolling without slipping of slab-like bodies such as cylinders, spheres or plane gears. The point of contact has an instantaneous zero velocity; there is a *pure rotation* about this point of contact. The body moves as if there is a hinge at the point of contact. For rolling without slipping, the acceleration of contact point is towards the geometric centre.

Thus, the equation $\mathbf{M}_A = \dot{\mathbf{H}}_A$ is valid in cases like the ones shown in Fig. 4.9a as *A* accelerates towards the centre of mass.

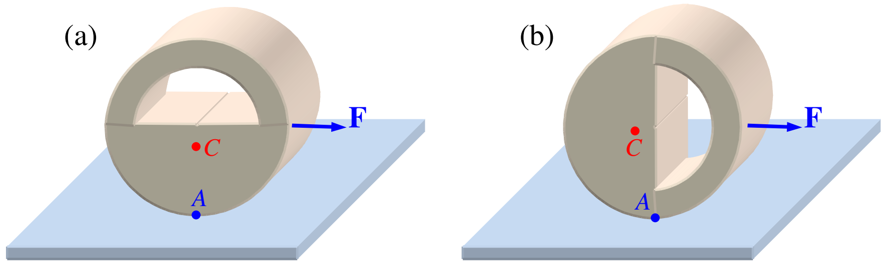

Figure 4.9

However, it is not valid for the case shown in Fig. 4.9b as A is *not* accelerating (we know that A is accelerating, at the instance vertical upward) toward mass centre (which is different from the geometric centre of the outer circle). Here we can use $M = I\alpha$ about the centre of mass.

General Plane Motion of a Slab-like Body

Consider the general plane motion of slab-like bodies. The motion is parallel to the plane of symmetry. The angular velocity $\boldsymbol{\omega}$ will be normal to the plane of symmetry and as per Chasles` theorem, can be taken to pass through the centre of mass. The transactional velocity $\mathbf{V}_c$ will be parallel to the plane of symmetry. Take XYZ as the inertial frame of reference and take xyz fixed at centre of mass such that xy-plane coincides with the plane of symmetry.

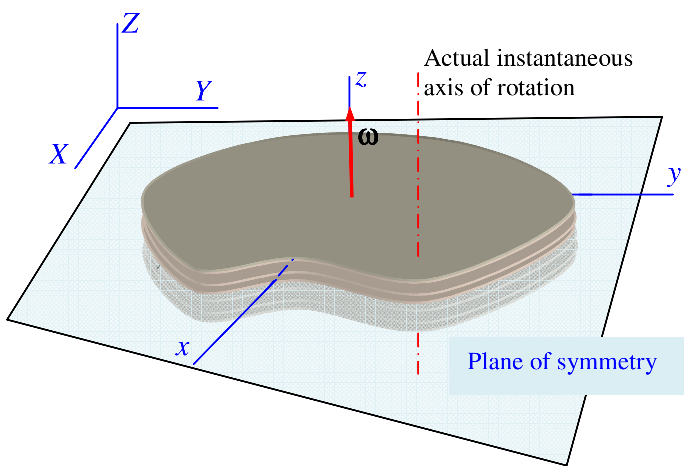

Figure 4.10

The moment of momentum equations are

$$M_x = 0,\ M_y = 0 \text{ and } M_z = I_{zz}\,\dot{\omega}\,.$$

Consider the centre of mass. For equilibrium in the z-direction, we have

$$\sum F_z = 0.$$

Newton's law holds in the x- and y-directions.

Ex: 4.3 Find the acceleration of block B shown in Fig. 4.11. The system is in a vertical plane and is released from rest. The cylinders roll without slipping along the vertical walls and along body B. Neglect friction along the guide rod. A torque of M_A = 150 Nm is applied to cylinder A. Other data are: W_A = 100 N, W_B = 300 N, W_c = 50 N, M_A = 150 Nm.

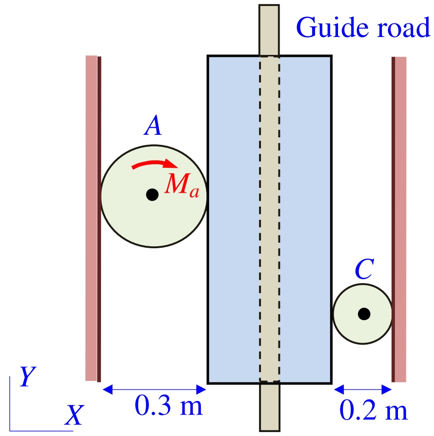

Figure 4.11

Consider the free body diagrams shown in Fig. 4.12. From the first free body diagram, the moment of momentum about *a*:(note that *a* accelerates towards the centre of mass)

$$-f_1(0.3)-100(0.15)-150=\left[\frac{1}{2}\frac{100}{g}(0.15)^2+\frac{100}{g}(0.15)^2\right]\ddot{\theta}_A,$$

which on simplification yields $-0.3f_1-165=0.344\ddot{\theta}_A$.

Considering the third free body diagram given by Fig. 4.12c, and writing the moment of momentum equation about *b*, we have

$$f_2(0.2)+50(0.1)=\left[\frac{1}{2}\frac{50}{g}(.1)^2+\frac{50}{g}(.1)^2\right]\ddot{\theta}_C,$$

which leads to $0.2f_2+5=0.07645\ddot{\theta}_C$.

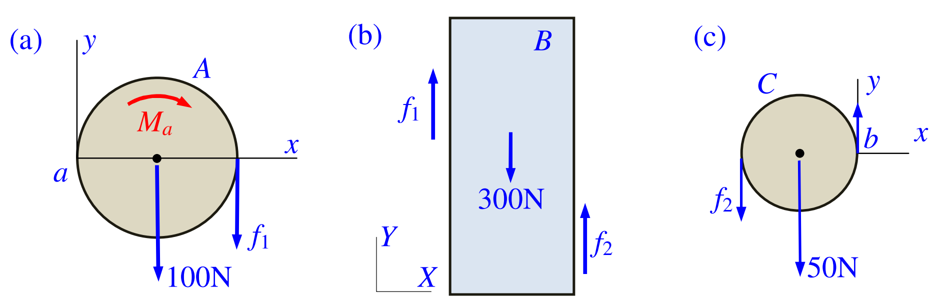

Figure 4.12

Writing the equation of motion along *y*-direction of the free body diagram given in Fig. 4.12b, we obtain

$$-300+f_1+f_2=\frac{300}{g}\ddot{y}_B.$$

From the kinematics of motion, we have

$$0.3\ddot{\theta}_A=\ddot{y}_B \text{ and } 0.2\ddot{\theta}_c=-\ddot{y}_B.$$

Solving the above equations simultaneously, we obtain

$$\ddot{y}_B = -24.1\,\text{m/s}^2 \text{ and } f_1 = -457.9\text{N} \text{ and } f_2 = 21.07\text{N}.$$

Pure Rotation of an Arbitrary Rigid Body

Consider a body having an arbitrary distribution of mass rotating about an axis of rotation fixed in inertial space. We consider this axis as the inertial Z-axis, and the z-axis fixed on the body. The origin of xyz can be anywhere along this axis of rotation. The moment of momentum equations will be the general equations given by Eq. (4.8), since I_{zx} and I_{yz} will, in general, not be equal to zero. If the centre of mass is along the z-axis, then it has no acceleration. Thus, we can apply the rules of statics to the centre of mass. For other cases, we need to employ Newton's law.

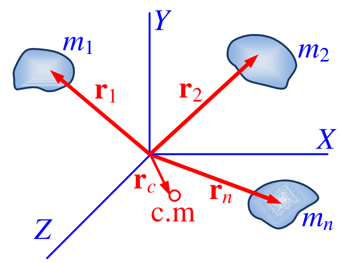

Figure 4.13

From the definition of centre of mass of a system of rigid bodies as shown in Fig. 4.13, we have

$$M\,\mathbf{r}_c = \sum_{i=1}^{n} m_i\,\mathbf{r}_i\ ,$$

where m_i is the mass of the i$^{\text{th}}$ rigid body, $\mathbf{r}_i$ is the position vector to the centre of mass of i$^{\text{th}}$ rigid body, M is the total mass of the system of rigid bodies, and $\mathbf{r}_c$ is the position vector of centre of mass of the system of rigid bodies. Differentiating this equation, we obtain

$$M\,\dot{\mathbf{r}}_c = \sum_i m_i\,\dot{\mathbf{r}}_i \text{ and } M\,\ddot{\mathbf{r}}_c = \sum_i m_i\,\ddot{\mathbf{r}}_i\ .$$

Euler's Equations

For the general motion of rigid bodies, the Euler's equations are obtained by orienting the xyz axes along the principal inertia directions. Thus, the inertia tensor boils down to a diagonal matrix. Eq. (4.6) for this case reduces to

$$M_x = I_{xx}\,\dot{\omega}_x + \omega_y\,\omega_z\,(I_{zz} - I_{yy}),$$

$$M_y = I_{yy}\,\dot{\omega}_y + \omega_z\,\omega_x\,(I_{xx} - I_{zz})$$

and

$$M_z = I_{zz}\,\dot{\omega}_z + \omega_x\,\omega_y\,(I_{yy} - I_{xx}).$$

These equations are nonlinear. However, if the motion of the body is known, we can easily compute the moments about point A (refer to the first few sections of this module).

Ex: 4.4 A thin disc of radius 1 m and weight 1.5 kN rotates at an angular speed ω_2 of 100 rad/s relative to a platform as shown in Fig. 4.14. The platform rotates with an angular speed ω_1 of 20 rad/s relative to ground. Calculate the reaction at the bearings A and B. Neglect the weight of the shaft. Assume that bearing A restrains the system in the radial direction.

Fix xyz to the centre of mass of the disc as shown. XYZ is fixed to ground.

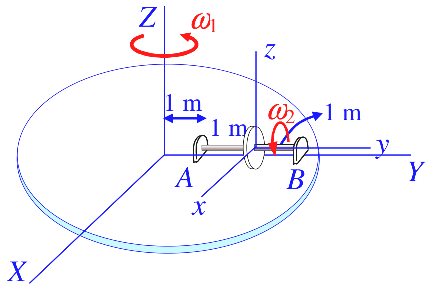

Figure 4.14

Angular velocity of disc with respect to XYZ is $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = 20\,\mathbf{k} + 100\,\mathbf{j}$ rad/s.

The xyz components of $\boldsymbol{\omega}$ are: $\omega_x = 0$, $\omega_y = 100$ rad/s, $\omega_z = 20$ rad/s.

Next, the angular accelerations are

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 = \mathbf{0} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 = 20\mathbf{k} \times 100\,\mathbf{j} = -2000\,\mathbf{i}\ \text{rad/s}^2.$$

$$I_{xx} = I_{zz} = \frac{M\,d^2}{16} = \frac{1500}{9.81} \times \frac{2^2}{16} = 38.226\ \text{kg/m}^2.\ I_{yy} = I_{xx} + I_{zz} = 76.452\ \text{kg/m}^2.$$

Therefore from Euler's equations

$$M_x = 38.226 \times (-2000) + 20 \times 100 \times (-38.226) = -152905.2\ \text{Nm};$$

and

$$M_y = 0 + 0 = 0 \text{ and } M_z = 0 + 0 = 0.$$

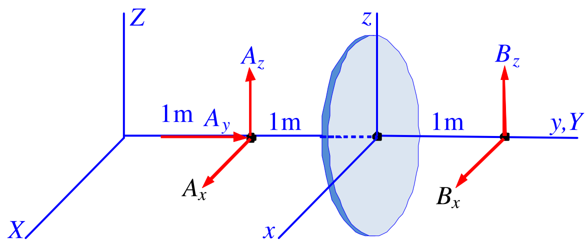

Figure 4.15

Now, consider the free body diagram of the shaft and disc shown in Fig. 4.15. The moments M_x, M_y and M_z are generated by the bearing forces. Hence, we can write

$$M_x = -152905.2 = 1 \times B_z - 1 \times A_z,$$

$$M_y = 0,$$

and

$$M_z = 0 = -1 \times B_x + 1 \times A_x.$$

From other equilibrium equations, we have

$$A_z + B_z = 1500\ \text{N},$$

$$A_x + B_x = 0$$

and

$$A_y = -\,(1500/9.81) \times 2 \times (20)^2 = \underline{-122{,}324.16}\ \text{N}.$$

And the remaining reactions work out as

$$A_x = B_x = 0,\ B_z = -\underline{75{,}702.6}\ \text{N},\ A_z = \underline{77{,}202.6}\ \text{N}.$$

All the four modules of the manuscript were typed and the fine figures drawn by Sajid, A.K.