ZZU102 ENGINEERING MECHANICS II—DYNAMICS

MODULE 3

Kinematics of Rigid Bodies—Relative Motion

We saw earlier the case of simple relative motion involving two references translating with respect to each other. There are many instances where the use of multiple references becomes inevitable. Recall that the Newton's laws are valid for an inertial reference only.

Translation and Rotation of Rigid Bodies

A rigid body is a continuum composed of particles having fixed distances between one another. There are two types of motion of a rigid body—translation and rotation.

TRANSLATION:

If a body moves in such a way that all the particles constituting it have at time t, the same velocity $V(t)$ relative to some reference, then the body is said to be in translation relative to this reference at this time. Translation does not necessarily mean motion along a straight line. A characteristic of translation is that a straight line such as ab drawn on the body retains an orientation parallel to its original direction throughout the motion of the body as shown in Fig. 3.1.

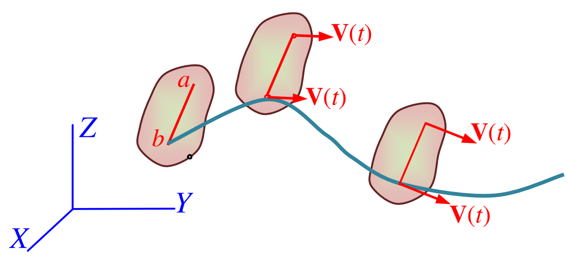

Figure 3.1

ROTATION:

If a rigid body moves such that along some straight line all the particles of the body, or a hypothetical extension of the body, have zero velocity relative to some reference, the body is said to be in rotation relative to this reference. This line of stationary particles is the *axis of rotation*.

MEASUREMENT OF ROTATION:

A single revolution is the amount of rotation in either a clockwise or a counter-clockwise sense about the axis of rotation that brings the body back to its original position. Partial revolutions are measured by observing any line segment such as AB, from a view point along the axis of rotation as depicted in Fig. 3.2. The angle β is the measure of the partial rotation.

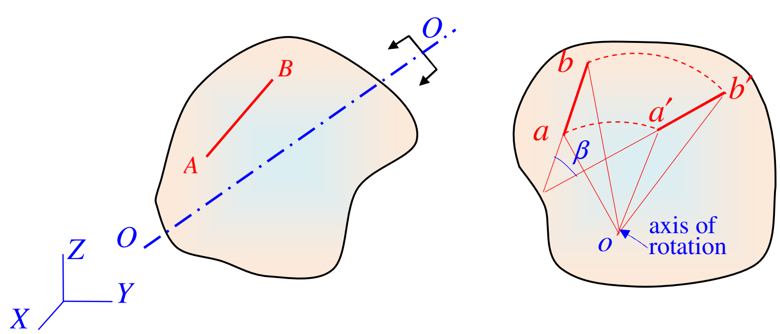

Figure 3.2

The reader may recall that finite rotations are *not* vectors (as their superposition is not commutative). However, infinitesimal rotations are vectors. Consequently, the angular velocity $\boldsymbol{\omega}$ having a magnitude $d\beta/dt$ with an orientation parallel to the axis of rotation (and sense according to the right hand screw rule) is a vector. Note also that the line of action is not prescribed by this definition (as the line of action can be considered at positions other than the axis of rotation as well).

Chasles` Theorem

The motion of any rigid body can be thought of as the superposition of a translational motion and a rotational motion.

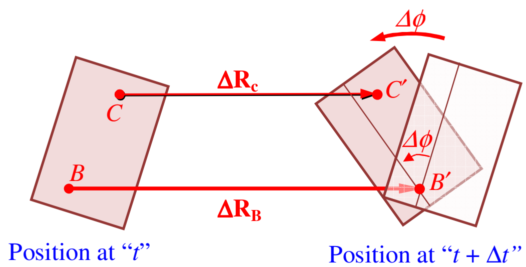

Figure 3.3

Consider the planar motion of a body as depicted in Fig. 3.3. Choose any point such as B which translates by $\Delta\mathbf{R_B}$ so that B reaches its final position B'. Now, apply a rotation about an axis through B (normal to the plane) by $\Delta\phi$. If we had chosen a different point, say C, the displacement vector $\Delta\mathbf{R_C}$ will differ from $\Delta\mathbf{R_B}$. However, the amount of rotation $\Delta\phi$ (about the axis through C) will be the same.

The translational velocity of the chosen point B at time t is $d\,\mathbf{R_B}/dt = \mathbf{V_B}$. The instantaneous angular velocity $\boldsymbol{\omega}$ is the same for any chosen point such as B. This holds good for any general motion of the body as well. This is known as the Chasles' theorem and can be described as follows.

CHASLES` THEOREM:

1. Select any point B in the body. Assume that all the particles of the body have the same velocity $\mathbf{V_B}$ at the time instant t, where $\mathbf{V_B}$ as the actual velocity of the point B.
2. Superpose a pure rotational velocity $\boldsymbol{\omega}$ about an axis of rotation going through point B.

With $\mathbf{V_B}$ and $\boldsymbol{\omega}$, the actual instantaneous motion of he body is completely determined; $\boldsymbol{\omega}$**,** will be the same for all the chosen points and only $\mathbf{V_B}$ will differ.

NOTE: The actual instantaneous axis of rotation at time t is the one going through those points of the body having zero velocity at time t.

Derivative of a Vector Fixed in a Moving Reference

Consider two references *XYZ* and *xyz*. We observe the moving frame of reference *xyz* from the inertial frame *XYZ*. As a reference is a rigid system, we apply Chasles` theorem to reference *xyz*. Thus, to fully describe the motion of *xyz* relative to *XYZ*, we chose the origin O, and we superpose a translation velocity $\dot{\mathbf{R}}$, equal to velocity of O onto a rotational velocity $\boldsymbol{\omega}$, with the axis of rotation passing through the point O. This is depicted in Fig. 3.4.

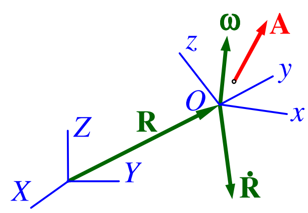

Figure 3.4

Now, suppose we have a vector **A** of *fixed length*, and *fixed orientation* as seen from *xyz*. Such a vector is said to be *fixed* in reference *xyz*. Then, we can write

$$\left(\frac{d\mathbf{A}}{dt}\right)_{xyz} = 0.$$

However,

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ}$$

may not be zero. To evaluate this, let us use Chasles` theorem as follows.

1. Consider the translational motion of *xyz* as given by $\dot{\mathbf{R}}$. This does not alter the direction of **A** as seen from *XYZ*. Moreover the magnitude of **A** is fixed, (although the line of action of **A** may change). As a result, the vector **A** does not change during this motion.
2. Next, consider a pure rotation about a stationary axis collinear with $\boldsymbol{\omega}$ and passing through *O*.

In order to observe this rotation, let us employ at *O* a stationary reference *X′Y′Z′* positioned in such a way that *Z′*-axis coincides with the axis of rotation. See Fig. 3.5 below.

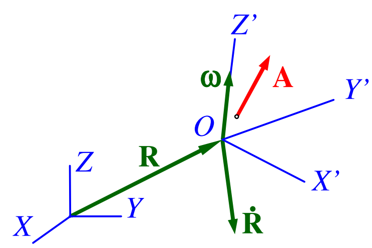

Figure 3.5

Now, the vector **A** is rotating about the *Z′*-axis at the instant *t*. We can write **A** as

$$\mathbf{A} = A_r\,\boldsymbol{\varepsilon}_r + A_\theta\,\boldsymbol{\varepsilon}_\theta + A_{Z'}\,\boldsymbol{\varepsilon}_{Z'}\,.$$

As **A** rotates about the *Z′*-axis, its components A_r, A_θ and $A_{Z'}$ do not change with time. Hence, we have

$$\left(\frac{d\mathbf{A}}{dt}\right)_{X'Y'Z'} = A_r\left(\frac{d\boldsymbol{\varepsilon}_r}{dt}\right)_{X'Y'Z'} + A_\theta\left(\frac{d\boldsymbol{\varepsilon}_\theta}{dt}\right)_{X'Y'Z'}.$$

We have seen earlier that

$$\frac{d\boldsymbol{\varepsilon}_r}{dt} = \dot{\theta}\,\boldsymbol{\varepsilon}_\theta = \omega\boldsymbol{\varepsilon}_\theta \text{ and } \frac{d\boldsymbol{\varepsilon}_\theta}{dt} = -\dot{\theta}\,\boldsymbol{\varepsilon}_r = -\omega\boldsymbol{\varepsilon}_r\,.$$

Thus, we obtain

$$\left(\frac{d\mathbf{A}}{dt}\right)_{X'Y'Z'} = A_r\,\omega\boldsymbol{\varepsilon}_\theta - A_\theta\,\omega\boldsymbol{\varepsilon}_r\,,$$

which can be shown to be equal to $\boldsymbol{\omega}\times\mathbf{A}$ as

$$\boldsymbol{\omega}\times\mathbf{A} = \begin{vmatrix} \boldsymbol{\varepsilon}_r & \boldsymbol{\varepsilon}_\theta & \boldsymbol{\varepsilon}_{Z'} \\ 0 & 0 & \omega \\ A_r & A_\theta & A_{Z'} \end{vmatrix} = -\omega\mathbf{A}_\theta\,\boldsymbol{\varepsilon}_r + \omega A_r\,\boldsymbol{\varepsilon}_\theta\,.$$

Therefore, we can write

$$\left(\frac{d\mathbf{A}}{dt}\right)_{X'Y'Z'} = \boldsymbol{\omega}\times\mathbf{A}.$$

Since $X'Y'Z'$ is stationary relative to XYZ, we would observe the same time derivative from the latter reference as from the former one. That is

$$\left(\frac{d}{dt}\right)_{X'Y'Z'} = \left(\frac{d}{dt}\right)_{XYZ}.$$

Thus, we conclude that

$$\boxed{\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \boldsymbol{\omega}\times\mathbf{A}}.$$

The above equation provides the time rate of change of **A** fixed in a moving reference *xyz* moving arbitrarily relative to *XYZ*. We see from the above that this time rate of change of **A** remains unchanged when

1. **A** is fixed in some other location (as long as its magnitude and direction are the same), and
2. The actual axis of rotation is shifted to a new parallel position.

Differentiating the above once again, we get

$$\left(\frac{d^2\mathbf{A}}{dt^2}\right)_{XYZ} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ}\times\mathbf{A} + \boldsymbol{\omega}\times\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \dot{\boldsymbol{\omega}}\times\mathbf{A} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A}),$$

where $\dot{\boldsymbol{\omega}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ}$.

NOTE:

1. The term "fixed in reference *xyz*" can be replaced by "fixed in a rigid body". Then $\boldsymbol{\omega}$ is the angular velocity of the rigid body.
2. Let the angular velocity of a body *A* relative to another body *B* be $\boldsymbol{\omega}_1$ and the angular velocity of *B* relative to the ground be $\boldsymbol{\omega}_2$. Then, what is the total angular velocity $\boldsymbol{\omega}_T$ of *A* relative to ground? Here $\boldsymbol{\omega}_1$ (the angular velocity of *A* relative to *B*) is actually the difference between the total angular velocity of *A* with respect to ground and the angular velocity $\boldsymbol{\omega}_2$ of *B* with respect to ground. i.e. $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_T - \boldsymbol{\omega}_2$. Therefore, $\boldsymbol{\omega}_T = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$.

Ex: 3.1 A disk *C* is mounted on a shaft *AB* as shown in Fig. 3.6. The shaft and the disk rotate at a constant angular velocity ω_2 of 10 rad/s, relative to the platform to which the bearings *A* and *B* are attached. The platform rotates at a constant angular speed ω_1 of 5 rad/s relative to the ground in a direction parallel to the *Z*-axis of the ground reference *XYZ*. Find the angular velocity $\boldsymbol{\omega}$ of the disk relative to *XYZ*. Find $(d\boldsymbol{\omega}/dt)_{XYZ}$ and $(d^2\boldsymbol{\omega}/dt^2)_{XYZ}$.

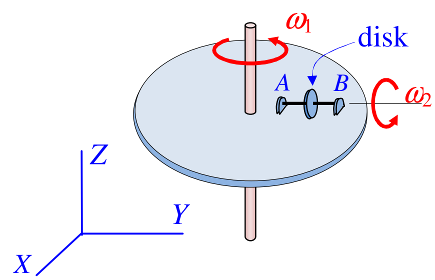

Figure 3.6

The angular velocity $\boldsymbol{\omega}$ of the disk relative to the ground is, $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. At the instance shown, $\boldsymbol{\omega} = 5\,\mathbf{k} + 10\,\mathbf{j}$ rad/s. Using a dot to represent time derivative with respect to *XYZ*, we have

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 .$$

Consider the vector $\boldsymbol{\omega}_2$. It is always collinear with *AB*. Moreover, $\boldsymbol{\omega}_2$ is a constant. Thus, $\boldsymbol{\omega}_2$ is *fixed* to the platform along *AB*. Since, the platform has an angular velocity of $\dot{\boldsymbol{\omega}}_1$ relative to *XYZ*, we have

$$\dot{\boldsymbol{\omega}}_2 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 .$$

Now, $\dot{\boldsymbol{\omega}}_1 = \mathbf{0}$ as $\dot{\boldsymbol{\omega}}_1$ as seen from *XYZ* is a constant vector. Hence, we have

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 = 5\mathbf{k} \times 10\mathbf{j} = -50\mathbf{i}\ \text{rad/s}^2 .$$

Taking derivative of the above once again, we get

$$\ddot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 \times \dot{\boldsymbol{\omega}}_2 = \mathbf{0} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) = 5\mathbf{k} \times (5\mathbf{k} \times 10\mathbf{j}) = -250\mathbf{j}\text{rad/s}^2 .$$

Ex: 3.2 Consider a position vector $\boldsymbol{\rho}$ between two points on the rotating disk of last example as shown in Fig. 3.7. The length of $\boldsymbol{\rho}$ is 100mm and, at the instant of interest, is in the vertical direction. What are the first and second derivatives of $\boldsymbol{\rho}$ at this instant as seen from the ground reference?

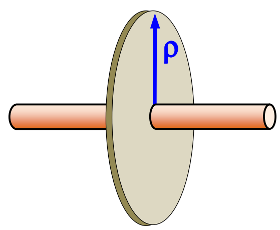

Figure 3.7

As the vector $\boldsymbol{\rho}$ is fixed to the disk which has, at all times, an angular velocity relative to *XYZ* equal to $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$, we have

$$\dot{\boldsymbol{\rho}} = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \boldsymbol{\rho} = (5\mathbf{k} + 10\mathbf{j}) \times 100\mathbf{k} = 1000\mathbf{i}\ \text{mm/s} .$$

Differentiating the above once again, we get

$$\ddot{\boldsymbol{\rho}} = \left(\frac{d}{dt}\dot{\boldsymbol{\rho}}\right)_{XYZ} = (\dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2) \times \boldsymbol{\rho} + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \dot{\boldsymbol{\rho}} .$$

Now, $\dot{\boldsymbol{\omega}}_1 = 0$ as seen earlier. Moreover, $\dot{\boldsymbol{\omega}}_2 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2$. Therefore, we have

$$\ddot{\boldsymbol{\rho}} = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) \times \boldsymbol{\rho} + (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) \times \dot{\boldsymbol{\rho}} = -50\mathbf{i} \times 100\mathbf{k} + (5\mathbf{k} + 10\mathbf{j}) \times 1000\mathbf{i}$$

$$= 5000\mathbf{j} + 5000\mathbf{j} - 10000\mathbf{k}\ \text{mm/s} = 10\mathbf{j} - 10\mathbf{k}\ \text{m/s} .$$

Ex: 3.3 Consider the same example as before shown in Fig. 3.8. For the disk, $\boldsymbol{\omega}_2 = 6\,\text{rad/s}$ and $\dot{\boldsymbol{\omega}}_2 = 2\,\text{rad/s}$, both relative to platform at the instant of interest. At this instant, $\boldsymbol{\omega}_1 = 2\,\text{rad/s}$ and $\dot{\boldsymbol{\omega}}_1 = -3\,\text{rad/s}^2$ for the platform relative to ground. Find the angular acceleration vector $\dot{\boldsymbol{\omega}}$ for the disk relative to the ground at the instant of interest, where $\boldsymbol{\omega}$ is the angular velocity of the disk relative to ground at all times.

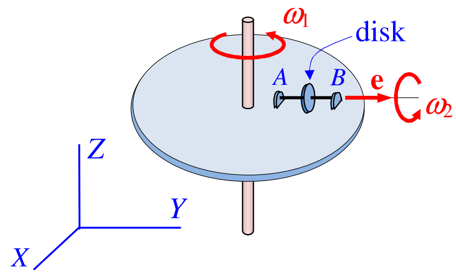

Figure 3.8

We have $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Therefore, $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2$. As $\boldsymbol{\omega}_1$ is vertical at all times, we have

$$\dot{\boldsymbol{\omega}}_1 = \left(\frac{d\boldsymbol{\omega}_1}{dt}\right)_{XYZ} = \dot{\omega}_1\,\mathbf{k} = -3\ \mathbf{k}\ \text{rad/s}^2.$$

On the other hand, $\boldsymbol{\omega}_2$ is changing direction and magnitude. Hence, $\boldsymbol{\omega}_2$ is *not* fixed in a reference or a rigid body. So, let us fix a unit vector **e** onto the platform as depicted in Fig. 3.8 to be collinear with the centre line of the shaft *AB*. The angular velocity of **e** is $\boldsymbol{\omega}_1$ at all times. Hence, $\boldsymbol{\omega}_2 = \omega_2\,\mathbf{e}$ at all times. Therefore,

$$\dot{\boldsymbol{\omega}}_2 = \dot{\omega}_2\,\mathbf{e} + \omega_2\,\dot{\mathbf{e}} = \dot{\omega}_2\,\mathbf{e} + \omega_2\,(\boldsymbol{\omega}_1 \times \mathbf{e}),$$

and

$$\dot{\boldsymbol{\omega}} = \dot{\omega}_1\,\mathbf{k} + \dot{\omega}_2\,\mathbf{e} + \omega_2\,(\boldsymbol{\omega}_1 \times \mathbf{e}).$$

This expression is valid at all times and we can differentiate it again. At the instant of interest, **e = j**. Thus, we obtain

$$\dot{\boldsymbol{\omega}} = -3\mathbf{k} + 2\mathbf{j} + 6(2\mathbf{k} \times \mathbf{j}) = -12\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\ \text{rad/s}^2.$$

Applications of the Fixed-Vector Concept

We saw that

$$\dot{\mathbf{A}} \equiv \left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \boldsymbol{\omega} \times \mathbf{A},$$

where $\boldsymbol{\omega}$ is the angular velocity of the body relative to *XYZ*. We shall use this formula for a vector $\boldsymbol{\rho}_{ab}$ as shown in Fig. 3.9 which connects two points *a* and *b* in a rigid body under consideration. The vector $\boldsymbol{\rho}_{ab}$ is *fixed* in the rigid body. As per Chasles' theorem, the body has a translational velocity $\dot{\mathbf{R}}$ relative to *XYZ* corresponding to some point *O* in the body plus an angular velocity $\boldsymbol{\omega}$ relative to *XYZ* with the axis of rotation through *O*.

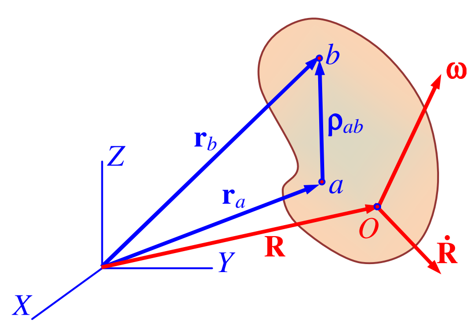

Figure 3.9

On observing $\boldsymbol{\rho}_{ab}$ from *XYZ*, we can see that $\dot{\boldsymbol{\rho}}_{ab} = \boldsymbol{\omega} \times \boldsymbol{\rho}_{ab}$.

Now, $\boldsymbol{\rho}_{ab} = \mathbf{r}_b - \mathbf{r}_a$. Hence, we have

$$\left(\frac{d\boldsymbol{\rho}_{ab}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{r}_b}{dt}\right)_{XYZ} - \left(\frac{d\,\mathbf{r}_a}{dt}\right)_{XYZ} = \mathbf{V}_b - \mathbf{V}_a,$$

where $\mathbf{V}_a$ and $\mathbf{V}_b$ are the velocities of the points a and b as shown in Fig. 3.9. Moreover, $\mathbf{V}_b - \mathbf{V}_a$ is the difference between the velocity of points b and a. Thus, we can write

$$\boxed{\mathbf{V}_b = \mathbf{V}_a + \boldsymbol{\omega} \times \boldsymbol{\rho}_{ab}}.$$

Note the order of a and b in $\boldsymbol{\rho}_{ab}$ above and remember that $\boldsymbol{\rho}_{ba} = -\boldsymbol{\rho}_{ab}$. Thus, the velocity of a particle b of a rigid body as seen from XYZ equals the velocity of any other particle a of the body as seen from XYZ plus the velocity of particle b relative to a ($= \boldsymbol{\omega} \times \boldsymbol{\rho}_{ab}$).

Differentiating the above equation once again, we get a relation connecting the acceleration vectors of two points of a rigid body as

$$\mathbf{a}_b = \mathbf{a}_a + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ} \times \boldsymbol{\rho}_{ab} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_{ab}),$$

which can be rewritten as

$$\boxed{\mathbf{a}_b = \mathbf{a}_a + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_{ab} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_{ab})}.$$

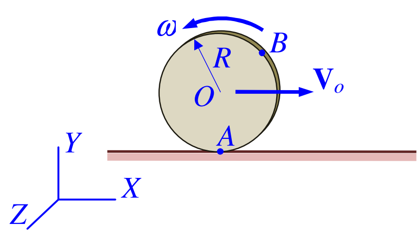

Figure 3.10

Consider a circular cylinder rolling without slipping as shown in Fig. 3.10. The point of contact A of the cylinder with the ground has zero velocity at the instant. Hence we have pure instantaneous rotation at any time t about an axis of rotation at the line of contact. The velocity of a point such as B shown in the figure can be obtained as

$$\mathbf{V}_B = V_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_{AB} = \mathbf{0} + \boldsymbol{\omega} \times \boldsymbol{\rho}_{AB} = \boldsymbol{\omega} \times \boldsymbol{\rho}_{AB}.$$

Thus, for computing the velocity of any point on the cylinder, we can imagine the cylinder as *hinged* at the point of contact A. For the point O, the centre, we get

$$\mathbf{V}_0 = \omega\mathbf{k} \times R\,\mathbf{j} = -\omega R\,\mathbf{i},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the coordinate directions X, Y and Z respectively. If V_0 is known, we then have

$$\omega = -\frac{V_0}{R}.$$

Differentiating $\mathbf{V}_0$ once we obtain

$$\mathbf{a}_0 = -R\dot{\omega}\mathbf{i} = -R\alpha\,\mathbf{i}.$$

The acceleration vector of the point of contact A can be calculated from

$$\mathbf{a}_A = \mathbf{a}_O + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_{OA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_{OA}) = -R\alpha\,\mathbf{i} + \dot{\omega}\mathbf{k} \times (-R\,\mathbf{j}) + \omega\mathbf{k} \times (\omega\mathbf{k} \times -R\,\mathbf{j}) = R\omega^2\,\mathbf{j}.$$

Thus, the point A is accelerating upward, toward the centre of the cylinder.

Ex: 3.4 Find the angular velocities and angular accelerations of the two bars shown in Fig. 3.11. The cylinder is rotating at a constant angular speed of 2 rad/s. Also locate the instantaneous axis of rotation of road AB.

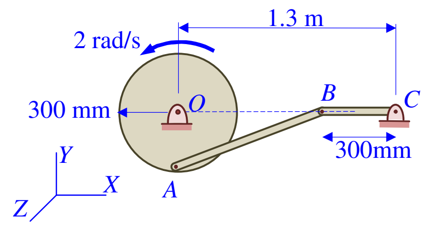

Figure 3.11

ANGULAR VELOCITIES:

Consider the cylinder first. Since $\mathbf{V}_O = \mathbf{0}$, we have

$$\mathbf{V}_A = \mathbf{V}_O + \boldsymbol{\omega} \times \boldsymbol{\rho}_{OA} = \mathbf{0} + 2\mathbf{k} \times (-0.3\mathbf{j}) = 0.6\mathbf{i} \text{ m/s}.$$

Now, let us consider the bar BC. We have

$$\mathbf{V}_B = \mathbf{V}_C + \boldsymbol{\omega}_{BC} \times \boldsymbol{\rho}_{CB} = \mathbf{0} + \omega_{BC}\mathbf{k} \times (-0.3\mathbf{i}) = -0.3\omega_{BC}\mathbf{j}$$

Considering bar AB next, we have

$$\mathbf{V}_B = \mathbf{V}_A + \boldsymbol{\omega}_{AB} \times \boldsymbol{\rho}_{AB} = 0.6\mathbf{i} + \omega_{AB}\mathbf{k} \times (1\mathbf{i} + 0.3\mathbf{j})$$

$$= 0.6\,\mathbf{i} + \omega_{AB}\,\mathbf{j} - 0.3\,\omega_{AB}\,\mathbf{i}.$$

Equating $\mathbf{V}_B$ from the above two, we obtain ω_{AB} = 2 rad/s and ω_{BC} = –6.667 rad/s. Thus ω_{AB} is counter-clockwise and ω_{BC} is clockwise.

ANGULAR ACCELERATIONS:

The acceleration of a point on a rigid body can be written as

$$\mathbf{a}_b = \mathbf{a}_a + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_{ab} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_{ab}).$$

The acceleration of point A can be obtained by considering the cylinder. Thus

$$\mathbf{a}_A = \mathbf{a}_O + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_{OA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_{OA}) = \mathbf{0} + \mathbf{0} + 2\mathbf{k} \times (2\mathbf{k} \times -0.3\mathbf{j}) = 1.2\,\mathbf{j} \text{ m/s}^2.$$

Similarly, considering the link BC, we get the acceleration of point B as follows:

$$\mathbf{a}_B = \mathbf{a}_C + \dot{\boldsymbol{\omega}}_{BC} \times \boldsymbol{\rho}_{CB} + \boldsymbol{\omega}_{BC} \times (\boldsymbol{\omega}_{BC} \times \boldsymbol{\rho}_{CB}) = \mathbf{0} + \dot{\omega}_{BC}\mathbf{k} \times (-0.3\mathbf{i}) + \omega_{BC}\mathbf{k} \times \{\omega_{BC}\mathbf{k} \times (-0.3\mathbf{i})\}$$

$$= 13.33\mathbf{i} - 0.3\dot{\omega}_{BC}\,\mathbf{j} \text{ m/s}^2$$

We can get the acceleration of point B from consideration of bar AB as well. Thus, we obtain

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\omega}}_{AB} \times \boldsymbol{\rho}_{AB} + \boldsymbol{\omega}_{AB} \times (\boldsymbol{\omega}_{AB} \times \boldsymbol{\rho}_{AB}) = 1.2\,\mathbf{j} + \dot{\omega}_{AB}\mathbf{k} \times (1\mathbf{i} + 0.3\mathbf{j}) + 2\mathbf{k} \times \{2\mathbf{k} \times (1\mathbf{i} + 0.3\mathbf{j})\}$$

$$= \dot{\omega}_{AB}\,\mathbf{j} - \dot{\omega}_{AB}\,\mathbf{i} - 4\,\mathbf{i}$$

Equating the above two, we get the angular accelerations as $\dot{\omega}_{AB}$ = –57.8 rad/s^2 and $\dot{\omega}_{BC}$ = 192.6 rad/s^2.

LOCATION OF INSTANTANEOUS AXIS OF ROTATION:

Consider Fig. 3.12. Let D (x, y) be the instantaneous axis of rotation. Then $\mathbf{V}_D$ must be equal to zero.

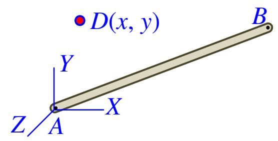

Figure 3.12

That is

$$\mathbf{V}_D = \mathbf{V}_A + \boldsymbol{\omega}_{AB} \times \boldsymbol{\rho}_{AD} = 0.6\mathbf{i} + 2\mathbf{k} \times (x\mathbf{i} + y\mathbf{j}) = 0.6\mathbf{i} + 2x\mathbf{j} - 2y\mathbf{i} = \mathbf{0},$$

from which we get $x = 0$ and $y = 0.3$ m. In other words, the point O of Fig. 3.11 happens to be the axis of rotation of bar AB at the instant of interest.

General Relationship between Time Derivatives of a Vector for Different References

We saw that for a vector **A** fixed in a moving reference *xyz*,

$$\left(\frac{d\mathbf{A}}{dt}\right)_{xyz} = \mathbf{0} \text{ and } \left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \boldsymbol{\omega} \times \mathbf{A}.$$

Let us extend the above to a vector **A** which is not necessarily fixed in reference *xyz*.

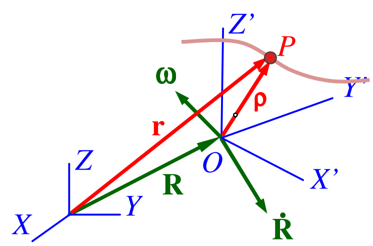

Figure 3.13

Consider particle P with position vector $\boldsymbol{\rho}$ with respect to *xyz* as shown in Fig. 3.13. Assume that the coordinates *xyz* moves arbitrarily relative to *XYZ*, with a translational velocity $\dot{\mathbf{R}}$ and a rotational velocity $\boldsymbol{\omega}$ in accordance with Chasles` theorem.

Let $\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ relative to *xyz*. Therefore we have

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}.$$

As $\dot{x}, \dot{y}, \dot{z}$ are time derivatives of scalars and so no reference need be specified for the time derivative. Now

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ} = (\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}) + (x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}}).$$

As **i**, **j**, **k** are unit vectors fixed in reference *xyz*, we have

$$x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}} = x(\boldsymbol{\omega} \times \mathbf{i}) + y(\boldsymbol{\omega} \times \mathbf{j}) + z(\boldsymbol{\omega} \times \mathbf{k}) = \boldsymbol{\omega} \times \boldsymbol{\rho}.$$

Hence,

$$\boxed{\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ} = \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} + \boldsymbol{\omega} \times \mathbf{A}.}$$

Here, $\boldsymbol{\omega}$ is the angular velocity of *xyz* relative to *XZY*.

Relationship between Velocities of a Particle for Different References

The velocity of a particle relative to a reference is the derivative as seen from this reference of the position vector of the particle in the reference. Thus

$$\mathbf{V}_{XYZ} = \left(\frac{d\mathbf{r}}{dt}\right)_{XYZ} \text{ and } \mathbf{V}_{xyz} = \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz}.$$

Now, $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$. Taking time derivatives, we get

$$\left(\frac{d\mathbf{r}}{dt}\right)_{XYZ} = \mathbf{V}_{XYZ} = \left(\frac{d\mathbf{R}}{dt}\right)_{XYZ} + \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ}.$$

Let $\dot{\mathbf{R}} \equiv \left(\frac{d\mathbf{R}}{dt}\right)_{XYZ}$ be the velocity of origin of xyz relative to XYZ. Therefore,

$$\mathbf{V}_{XYZ} = \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} + \dot{\mathbf{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \text{ or } \boxed{\mathbf{V}_{XYZ} = \mathbf{V}_{xyz} + \dot{\mathbf{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho}}.$$

The above relates the velocities of the same particles from the same references. This multi reference approach has great particle significance. We shall use a dot over a vector to indicate derivative relative to *XYZ*.

Acceleration of a Particle for Different References

From first principles, we can write

$$\mathbf{a}_{xyz} = \left(\frac{d}{dt}\mathbf{V}_{xyz}\right)_{xyz} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{xyz}$$

and

$$\mathbf{a}_{XYZ} = \left(\frac{d}{dt}\mathbf{V}_{XYZ}\right)_{XYZ} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{XYZ}.$$

Differentiating the velocity equation that we derived in the last section with respect to time relative to *XYZ* we obtain

$$\mathbf{a}_{XYZ} = \left(\frac{d}{dt}\mathbf{V}_{xyz}\right)_{XYZ} + \ddot{\mathbf{R}} + \left[\frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{\rho})\right]_{XYZ} = \left(\frac{d}{dt}\mathbf{V}_{xyz}\right)_{XYZ} + \ddot{\mathbf{R}} + \boldsymbol{\omega} \times \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ} + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ} \times \boldsymbol{\rho}.$$

However,

$$\left(\frac{d}{dt}\mathbf{V}_{xyz}\right)_{XYZ} = \left(\frac{d}{dt}\mathbf{V}_{xyz}\right)_{xyz} + \boldsymbol{\omega} \times \mathbf{V}_{xyz} \text{ and } \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ} = \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} + \boldsymbol{\omega} \times \boldsymbol{\rho}.$$

Hence,

$$\mathbf{a}_{XYZ} = \mathbf{a}_{xyz} + \boldsymbol{\omega} \times \mathbf{V}_{xyz} + \ddot{\mathbf{R}} + \boldsymbol{\omega} \times \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{XYZ} \times \boldsymbol{\rho}.$$

That is

$$\boxed{\mathbf{a}_{XYZ} = \mathbf{a}_{xyz} + \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \times \mathbf{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})},$$

where $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are the angular velocity and angular acceleration of reference *xyz* relative to *XYZ* respectively. The term $2\,\boldsymbol{\omega}\times\mathbf{V}_{xyz}$ is known as the Coriolis acceleration vector.

Ex: 3.5 An airplane moving at 70 m/s is undergoing a roll at 2 rad/min. When the plane is horizontal, an antenna is moving at a speed of 2.5 m/s relative to the plane, and is at a position of 3 m from the centre line of the plane. If the axis of roll is along the centreline, what is the velocity of antenna end relative to ground when the plane is horizontal?
Fix *xyz* to the airplane and *XYZ* to the ground.

A. MOTION OF PARTICLE RELATIVE TO *XYZ*:

The position vector of antenna tip relative to *xyz* is $\boldsymbol{\rho} = 3\,\mathbf{j}\,\text{m}$ and its velocity, again relative to *xyz* is $\mathbf{V}_{xyz} = 2.5\,\mathbf{j}\,\text{m/s}$.

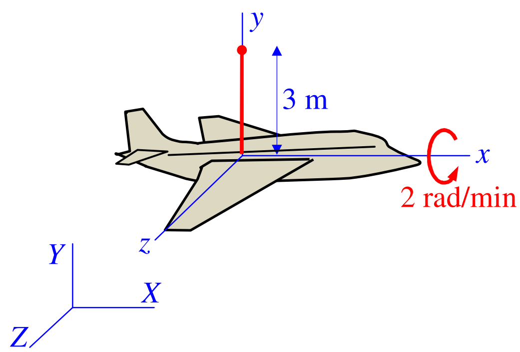

Figure 3.14

B. MOTION OF *XYZ* RELATIVE TO *XYZ*:

The translational velocity of *xyz* relative to *XYZ* is $\dot{\mathbf{R}} = 70\mathbf{i}\,\text{m/s}$ and its angular velocity is $\boldsymbol{\omega} = -\frac{2}{60}\mathbf{i}\,\text{rad/s}$.

Therefore, the velocity of the antenna tip relative to *XYZ* works out as

$$\mathbf{V}_{XYZ} = \mathbf{V}_{xyz} + \dot{\mathbf{R}} + \boldsymbol{\omega}\times\boldsymbol{\rho} = 2.5\,\mathbf{j} + 70\mathbf{i} + \left(-\frac{1}{30}\mathbf{i}\right)\times 3\,\mathbf{j} = 70\mathbf{i} + 2.5\,\mathbf{j} - 0.1\mathbf{k}\ \text{m/s}.$$

Ex: 3.6 A particle rotates at a constant angular speed of 10 rad/s on a platform, while the platform rotates with a constant angular speed of 50 rad/s about axis *AA*. What is the velocity of the particle *P* at the instant the platform is in the *XY* plane and the radius vector to the particle forms an angle of 30° with the *Y*-axis as shown in Fig. 3.15?

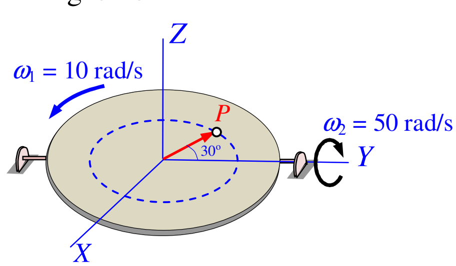

Figure 3.15

Fix *xyz* to the platform.

A. MOTION OF PARTICLE RELATIVE TO *XYZ*:

The position vector of particle *P* relative to *xyz* is $\boldsymbol{\rho} = 2(\cos 30\,\mathbf{j} + \sin 30\,\mathbf{i})\,\text{ft}$ and its velocity, again relative to *xyz* is $\mathbf{V}_{xyz} = \boldsymbol{\omega}_1 \times \boldsymbol{\rho} = 10\mathbf{k}\times(2\cos 30\,\mathbf{j} + \sin 30\,\mathbf{i}) = -17.3205\mathbf{i} - 10\,\mathbf{j}\,\text{ft/s}$.

B. Motion of *xyz* relative to *XYZ*:

The translational velocity of *xyz* relative to *XYZ* is $\dot{\mathbf{R}} = \mathbf{0}$ and its angular velocity is $\boldsymbol{\omega} = -50\,\mathbf{j}$ rad/s.

Therefore, the velocity of *P* relative to *XYZ* can be obtained as

$$\mathbf{V}_{XYZ} = \mathbf{V}_{xyz} + \dot{\mathbf{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} = -17.3205\,\mathbf{i} - 10\,\mathbf{j} + (-50\,\mathbf{j}) \times (2\cos 30\,\mathbf{j} + \sin 30\,\mathbf{i})\,.$$

$$= -17.3205\,\mathbf{i} - 10\,\mathbf{j} - 50\,\mathbf{k} \text{ ft/s}\,.$$

Ex: 3.7 A stationary truck as shown in Fig. 3.16 is carrying a cockpit for a worker who repairs overhead fixtures. At the instant shown the base *D* is rotating at $\boldsymbol{\omega}_2$ = 0.1 rad/s, and $\dot{\boldsymbol{\omega}}_2$ =0.2 rad/s^2 relative to the truck. Arm *AB* is rotating at angular speed of $\boldsymbol{\omega}_1$ = 0.2 rad/s and $\dot{\boldsymbol{\omega}}_1$ = 0.8 rad/s^2 relative to *DA*. Cockpit *C* is rotating relative to *AB* so as to keep the mass always upright. What are the velocity and acceleration vectors of the man relative to the ground if $\alpha = 45^{\circ}$ and $\beta = 30$ at the instant of interest? *DA*=13 m.

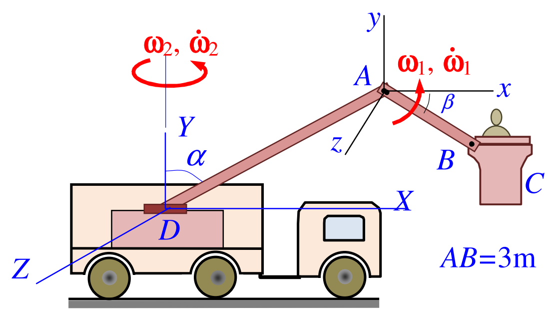

Figure 3.16

The cockpit *C* and the poin`t *B* have the same motion (as cockpit has the same orientation). So we will concentrate on *B* instead of *C*. Let us fix *xyz* to the arm *DA* and *XYZ* to the truck.

A. Motion of *B* relative to *xyz*:

The position vector of *B* relative to *xyz* is given by

$$\boldsymbol{\rho} = 3\cos\beta\,\mathbf{i} - 3\sin\beta\,\mathbf{j} = 2.60\,\mathbf{i} - 1.5\,\mathbf{j}\,\text{m}$$

As $\boldsymbol{\rho}$ is fixed in *AB* which rotates at $\boldsymbol{\omega}_1 = 0.2\,\mathbf{k}$, $\mathbf{V}_{xyz} = \boldsymbol{\omega} \times \boldsymbol{\rho} = 0.520\,\mathbf{j} + 0.3$ m/s.

$$\mathbf{a}_{xyz} = \left(\frac{d\boldsymbol{\omega}_1}{dt}\right)_{xyz} \times \boldsymbol{\rho} + \boldsymbol{\omega}_1 \times \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz}$$

$$= 0.8\,\mathbf{k} \times (2.60\,\mathbf{i} - 1.5\,\mathbf{j}) + 0.2\,\mathbf{k} \times (0.520\,\mathbf{j} + 0.3\,\mathbf{i}) = 1.09\,\mathbf{i} + 2.14\,\mathbf{j}\,\text{m/s}^2\,.$$

B. Motion of *xyz* relative to *XYZ*:

The position vector of the origin of *xyz* relative to *XYZ* is

$$\mathbf{R} = 13\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = 9.1923\,\mathbf{i} + 9.1923\,\mathbf{j}$$

As **R** is fixed in *DA*, and as *DA* rotates with $\boldsymbol{\omega}_2$ with respect to *XYZ*, we have

$$\dot{\mathbf{R}} = \boldsymbol{\omega}_2 \times \mathbf{R} = 0.1\,\mathbf{j} \times \mathbf{R} = -0.919\,\mathbf{k}\,\text{m/s}\,.$$

$$\ddot{\mathbf{R}} = \dot{\boldsymbol{\omega}}_2 \times \mathbf{R} + \boldsymbol{\omega}_2 \times \dot{\mathbf{R}} = 0.2\,\mathbf{j} \times (9.1923\,\mathbf{i} + 9.1923\,\mathbf{j}) + 0.1\,\mathbf{j} \times (-0.919\,\mathbf{k}) = -1.838\,\mathbf{k} - 0.0919\,\mathbf{i}\,\text{m/s}^2\,.$$

Now, as

$$\boldsymbol{\omega} = \boldsymbol{\omega}_2 = 0.1\,\mathbf{j}\,\text{rad/s} \;\text{ and }\; \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_2 = 0.2\,\mathbf{j}\,\text{rad/s}^2\,,$$

we have

$$\begin{aligned}\mathbf{V}_{XYZ} &= \mathbf{V}_{xyz} + \dot{\mathbf{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \\ &= (0.520\mathbf{j} + 0.3\mathbf{i}) + (-0.919\mathbf{k}) + 0.1\mathbf{j} \times (2.60\mathbf{i} - 1.5\mathbf{j}) = 0.3\mathbf{i} + 0.52\mathbf{j} - 1.179\mathbf{k}\ \text{m/s}.\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{XYZ} &= \mathbf{a}_{xyz} + \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \times \mathbf{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \\ &= (1.09\mathbf{i} + 2.14\mathbf{j}) + (-1.838\mathbf{k} - 0.0919\mathbf{i}) + 2(0.1\mathbf{j}) \times (0.520\mathbf{j} + 0.3\mathbf{i}) + 0.1\mathbf{j} \times \{0.1\mathbf{j} \times (2.60\mathbf{i} - 1.5\mathbf{j})\} \\ &= 0.978\mathbf{i} + 2.14\mathbf{j} - 2.42\mathbf{k}\ \text{m/s}^2.\end{aligned}$$

Brief Recap: Kinematics of Rigid Bodies

For a fixed vector **A** (fixed in xyz),

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \boldsymbol{\omega} \times \mathbf{A} \quad \text{and} \quad \left(\frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}t^2}\right)_{XYZ} = \dot{\boldsymbol{\omega}} \times \mathbf{A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{A})$$

Application of fixed vector:

$$\mathbf{V}_b = \mathbf{V}_a + \boldsymbol{\omega} \times \boldsymbol{\rho}_{ab}$$

and

$$\mathbf{a}_a = \mathbf{a}_a + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_{ab} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_{ab})$$

General relationship:

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{A}}{dt}\right)_{xyz} + \boldsymbol{\omega} \times \mathbf{A}.$$

Relationship between velocities:

$$\mathbf{V}_{XYZ} = \mathbf{V}_{xyz} + \dot{\mathbf{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho}.$$

And lastly, the relationship between the accelerations

$$\boxed{\mathbf{a}_{XYZ} = \mathbf{a}_{xyz} + \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \times \mathbf{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})}$$
