Engineering Mechanics

Continued…(6)

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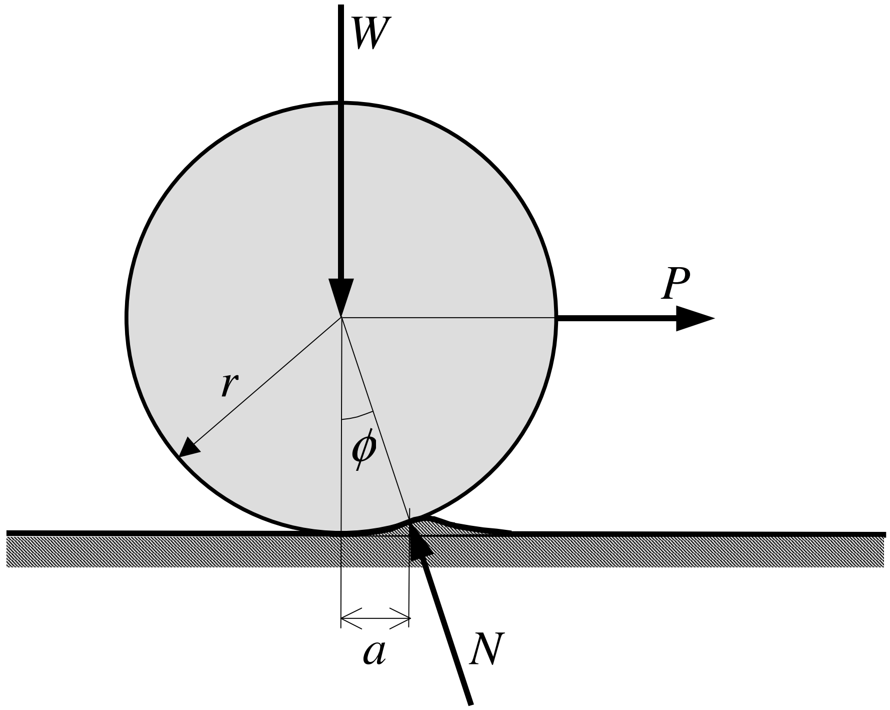
Rolling Resistance



Consider a hard roller, supporting a load *W* at the centre, moving without slipping along a horizontal surface.

A force *P* is needed to maintain uniform motion.

This can be explained by considering the deformation of the surface.

The reaction *N* is thus oriented at any angle α.

$$W = N\cos\phi \qquad P = N\sin\phi$$

$$\therefore \quad \tan\phi = \frac{P}{W}$$

As ϕ is small, $\tan\phi \approx \sin\phi \approx \frac{a}{r}$

$$\therefore \quad \frac{P}{W} = \frac{a}{r} \qquad or \quad P = \frac{Wa}{r}$$

Coulomb suggested that "*a*" depends on the materials, irrespective of *W* and *r*.

There are other opinions too in this regard.

<u>Coef. of Rolling Resistance (a mm)</u>	
Steel on steel	0.18 – 0.38
Steel on wood	1.52 – 2.54
Tyre on smooth road	0.50 – 0.76
Tyre on mud road	1.00 – 1.50
Hardened steel on h.s	0.005 – 0.01

Example: What is the rolling resistance of a railway coach weighing 1500 kN? The wheels are of 750 mm diameter, and the coefficient of rolling resistance between the wheel and the rail is 0.025 mm.

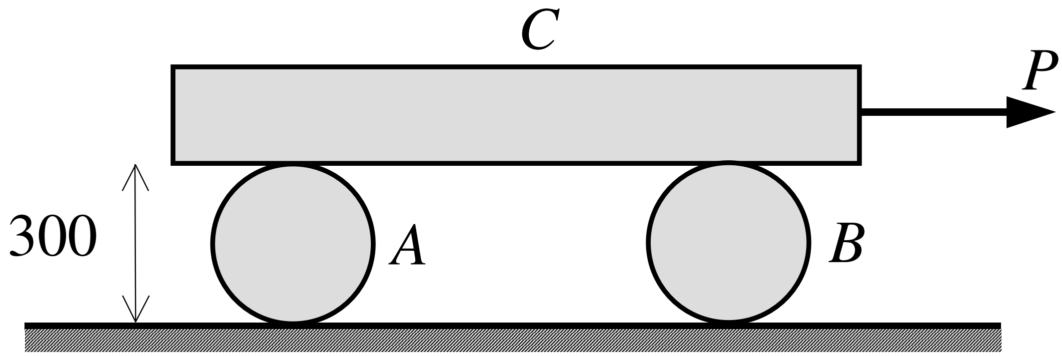
$$* \; P = \frac{Wa}{r} = \frac{1500 \times 0.025}{750} = 0.05 \text{ kN}$$

$$= 50 \text{ N}$$

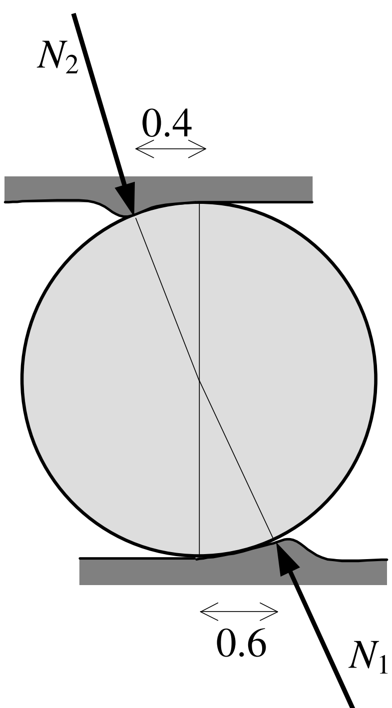
If it were a truck with the same weight, what is the value of the rolling resistance? The diameter of the tyres are 1.2 m, and *a* = 0.62 mm.

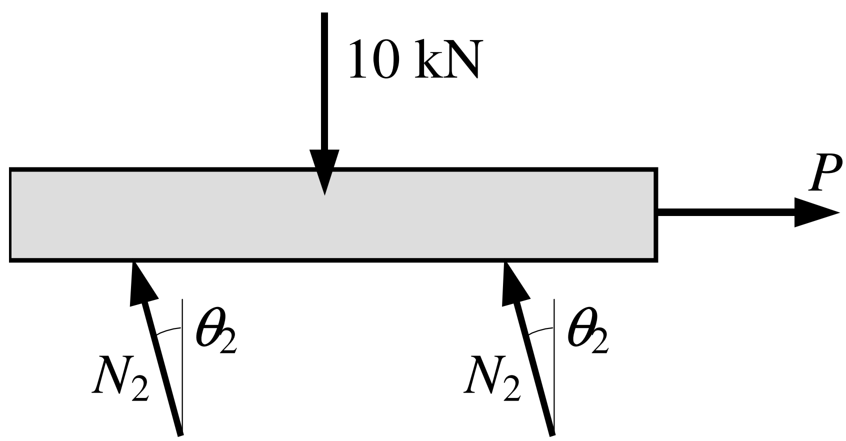
$$P = \frac{Wa}{r} = \frac{1500 \times 0.62}{1200} \times 1000 = 775 \text{ N}$$

(* the number of wheels has no influence; we divide by *n* and then multiply by it again).



Example: A block *C* weighing 10 kN is being moved on rollers *A* and *B*, each weighing 1 kN. What force *P* is needed to maintain steady motion? The coefficient of rolling resistance between the rollers and the ground is 0.6 mm, between block *C* and the rollers is 0.4 mm.

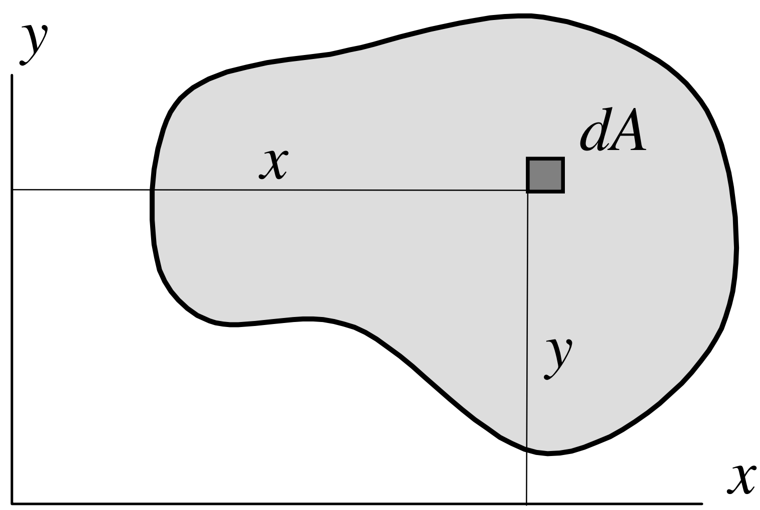




Properties of Surfaces

A variety of quantitative descriptions of surfaces are necessary in engineering work.

First Moment of Area and the Centroid



The first moment of a coplanar surface of area *A* about the *x*-axis is defined as

$$M_x = \int_A y \, dA$$

Similarly, the first moment of the area about *y*-axis is

$$M_y = \int_A x \, dA$$

These two quantities M_x and M_y convey a certain idea about the shape, size and orientation of the area which is useful in mechanics.

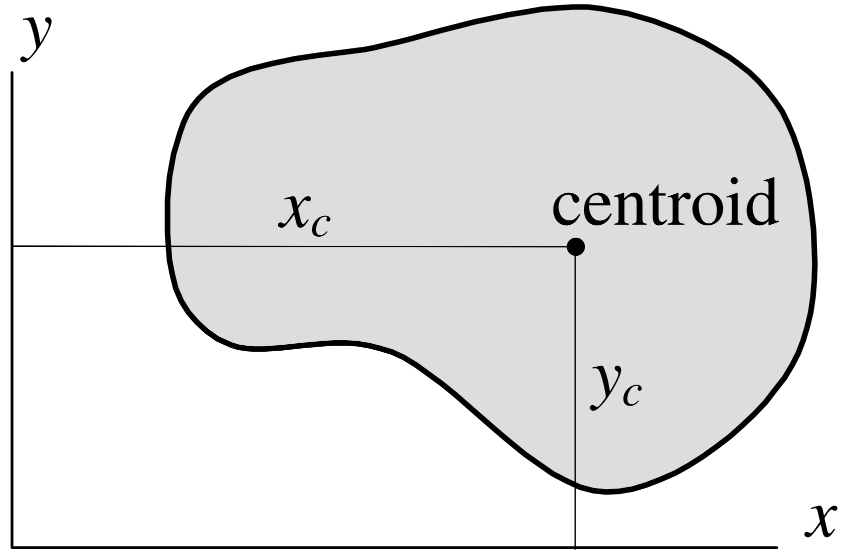
We can notice the similarity of this with the case of a distributed parallel force system.

In that case, we could replace the force system by a single resultant force located at a particular point $(\bar{x}, \bar{y})$.

Likewise, we can imagine the entire area to be concentrated at a single point called the *centroid* with the coordinates (x_c, y_c).

To compute these coordinates, we equate the moments of the distributed area with that of the concentrated area about both the axes.

Thus,



Therefore,

$$A\,y_c = \int_A y\,dA$$

$$y_c = \frac{\int_A y\,dA}{A} = \frac{M_x}{A}$$

Similarly,

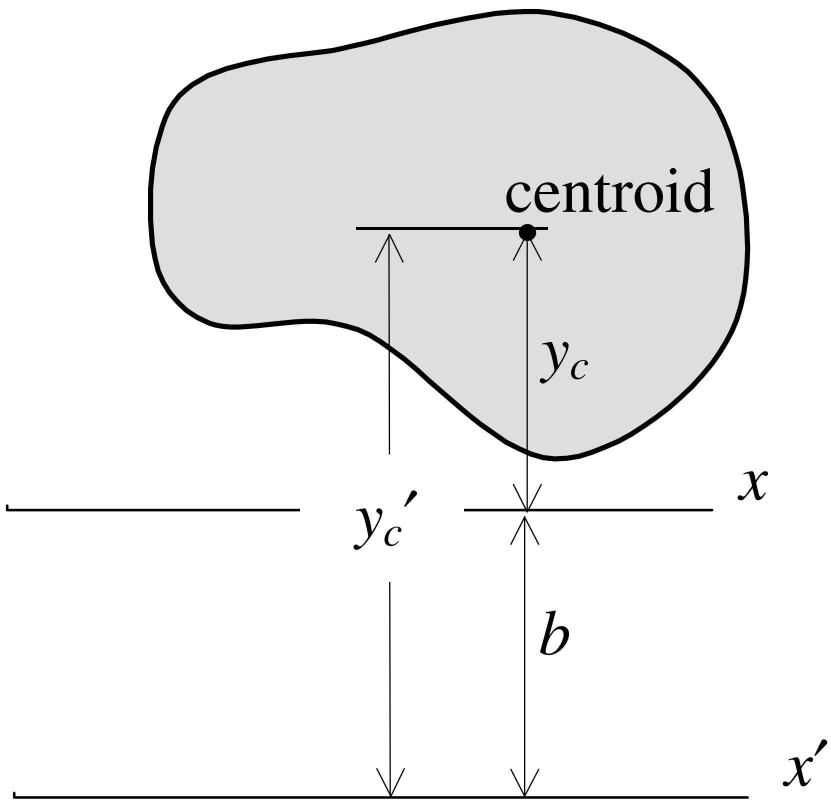
$$A\,x_c = \int_A x\,dA$$

and

$$x_c = \frac{\int_A x\,dA}{A} = \frac{M_y}{A}$$

The location of centroid of an area is independent of the location of the reference axes.

That is, the centroid is a property only of the area itself.



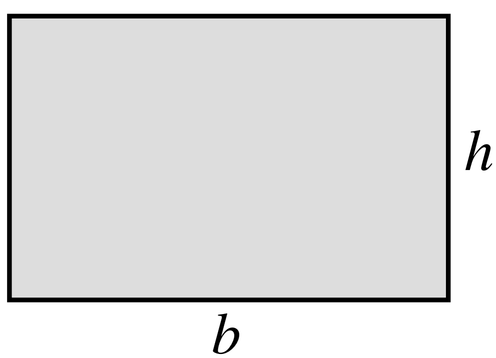
$$y_c' = \frac{\int_A (y+b)\,dA}{A} = \frac{M_x}{A} + b = y_c + b$$

All axes passing through the centroid are called *centroidal axes.*

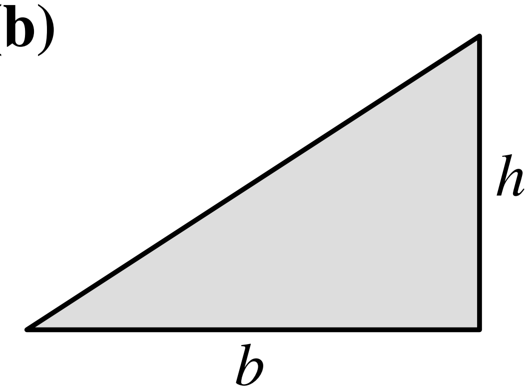
The first moment of an area about any of its centroidal axes is zero.

<u>Examples</u>: Determine the centroid of the following areas:

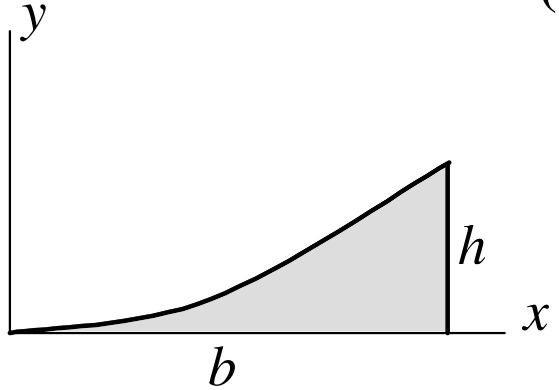
(a)



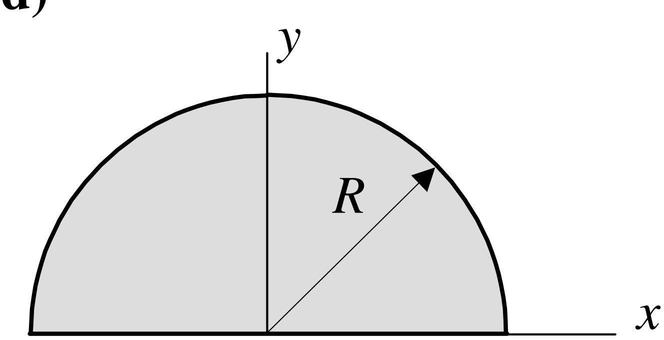
(b)



(c)

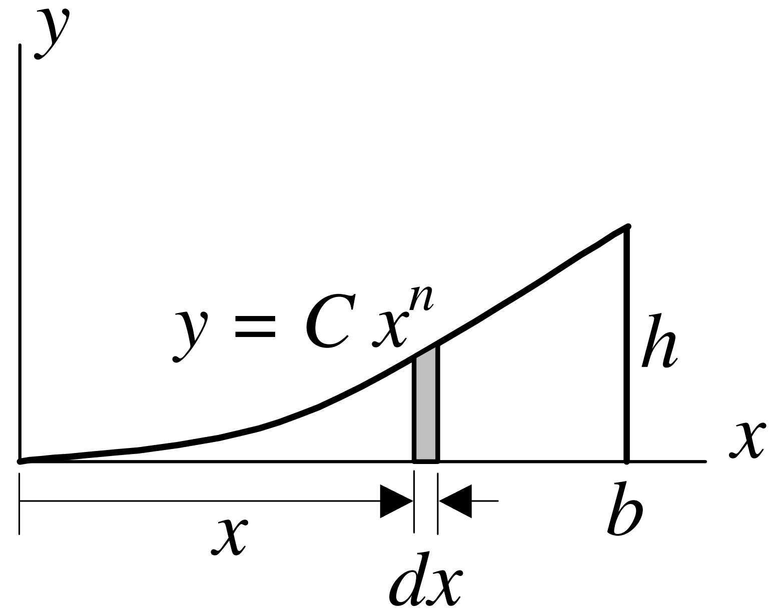


(d)



(c)

At $x = b$, $y = h$.
Hence, $C = h/b^n$



$$A = \int_0^b y\,dx = \int_0^b \frac{h}{b^n} x^n\,dx = \underline{\underline{\frac{bh}{n+1}}}$$

$$A\,\bar{x} = \int_0^b x\,y\,dx = \int_0^b \frac{h}{b^n} x^{n+1}\,dx = \frac{b^2 h}{(n+2)}$$

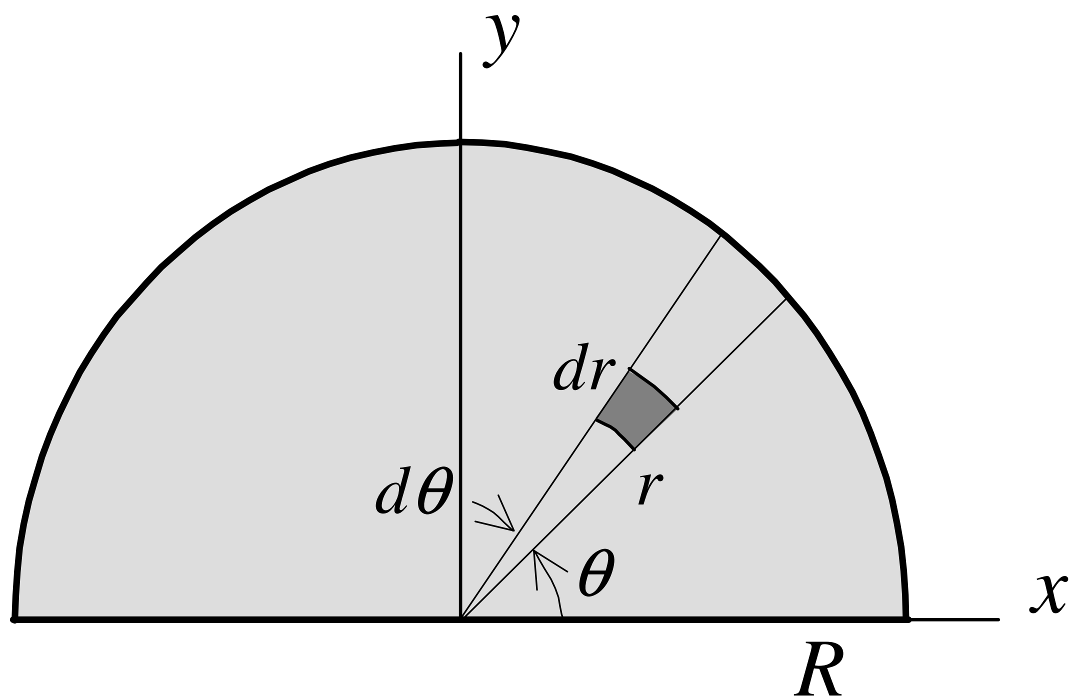
$$\therefore\ \ \bar{x} = \frac{b(n+1)}{(n+2)}$$

For a rectangle: $n = 0$

For a triangle: $n = 1$

For a parabola: $n = 2$

(d)

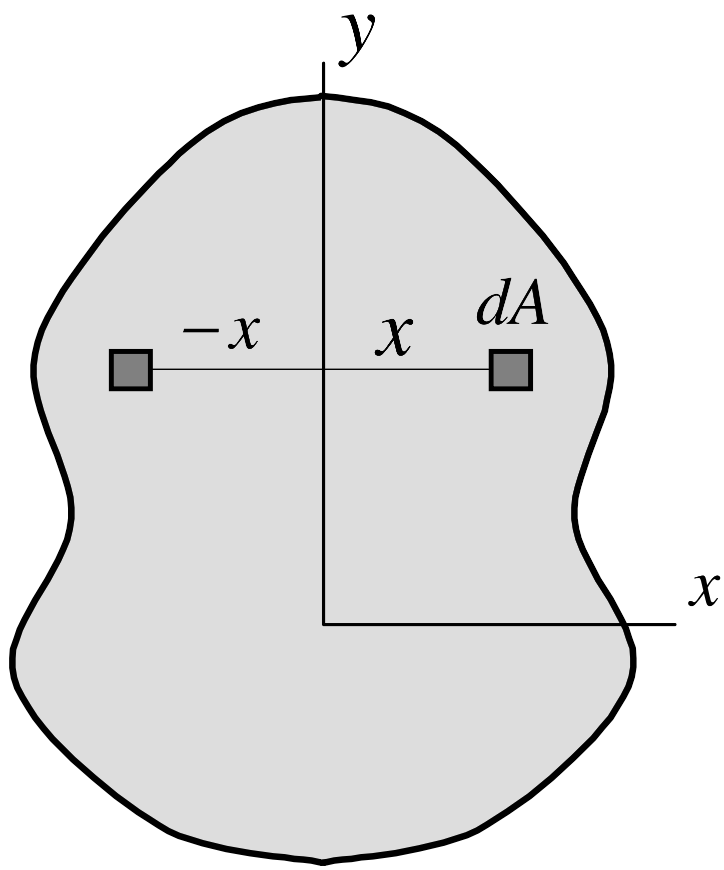


$$A = \int_{r=0}^{R} \int_{\theta=0}^{\pi} r\, dr\, d\theta = \frac{\pi R^2}{2}$$

$$A\, \bar{y} = \int_{r=0}^{R} \int_{\theta=0}^{\pi} r\, dr\, d\theta\, (r \sin\theta) = \frac{2 R^3}{3}$$

$$\therefore \quad \bar{y} = \frac{4R}{3\pi}$$

Area with One Axis of Symmetry



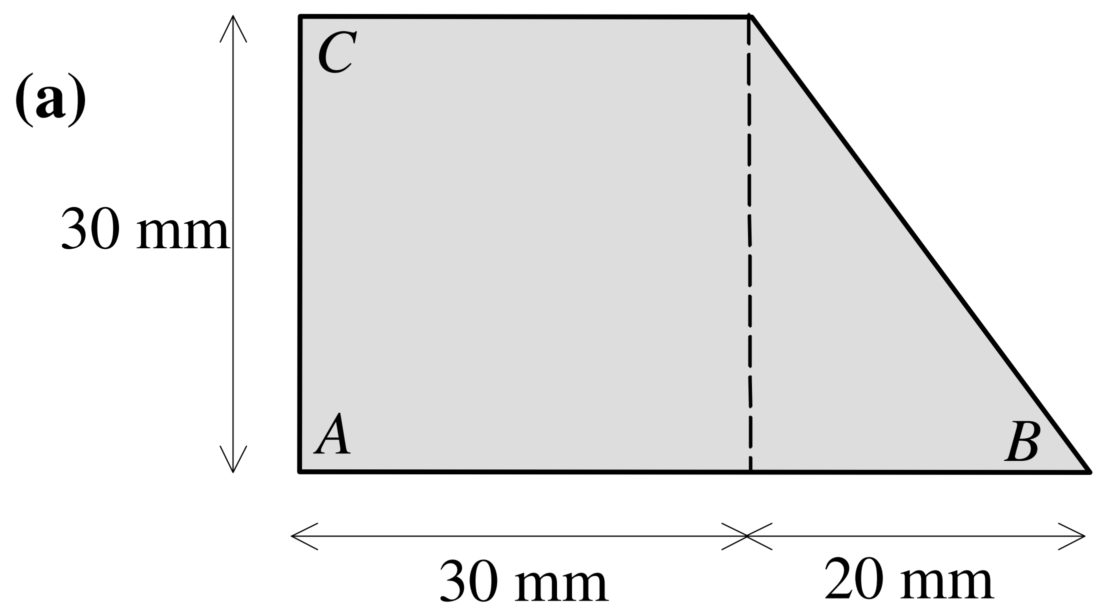
$$A\,x_c = \int_A x\,dA = 0$$

(as for every *+x dA*, there exists a *–x dA*)

Hence, the centroid must lie on the axis of symmetry.

Composite Areas

Example: Determine the centroid of the following area:



Area A = $30^2 + 30 \times 20/2$ = $\underline{\underline{1200 \text{ mm}^2}}$

Taking moment about AB,

$$A\bar{y} = 30 \times 30 \times 15 + \tfrac{1}{2} \times (30 \times 20) \times \tfrac{1}{3}(30)$$

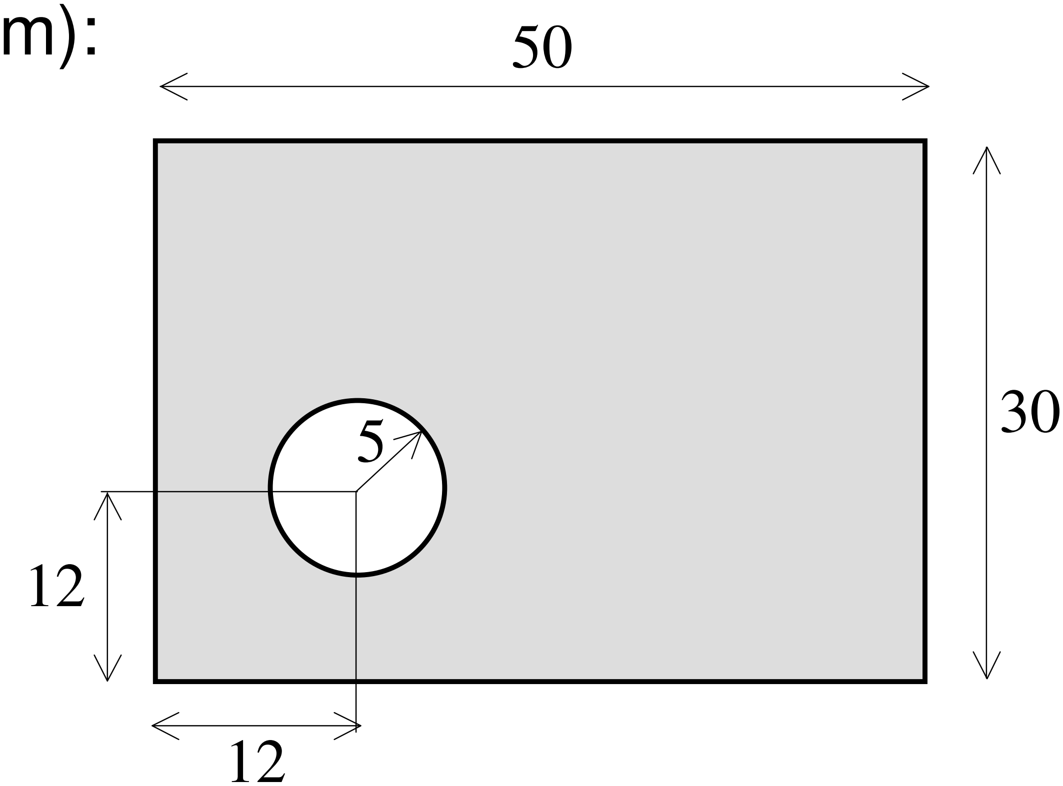
$$\therefore \ \bar{y} = 16{,}500 \div 1200 = \underline{\underline{13.75 \text{ mm}}}$$

Similarly, taking moments about AC

$$A\bar{x} = 30 \times 30 \times 15 + \tfrac{1}{2} \times (30 \times 20) \times [30 + \tfrac{1}{3}(20)]$$

$$\therefore \ \bar{x} = 24{,}500 \div 1200 = \underline{\underline{20.4167 \text{ mm}}}$$

Example: Determine the centroid of the following area (all dimensions shown are in mm):



Area,

$$A = 50 \times 30 - \frac{\pi 10^2}{4} = \underline{\underline{1421.46\,\text{mm}^2}}$$

Taking moments of area about the bottom edge

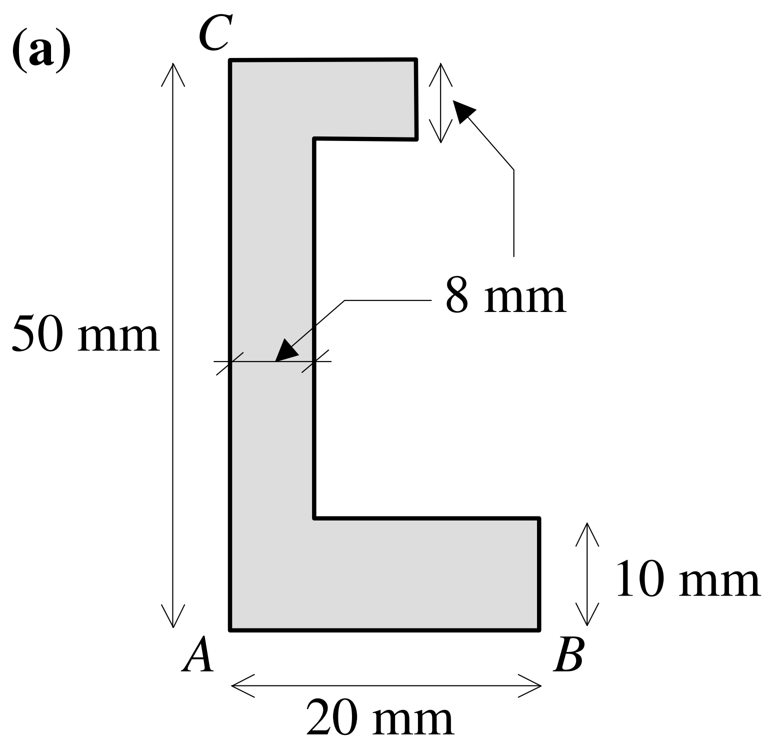
$$A\bar{y} = 50 \times 30 \times 15 - \frac{\pi 10^2}{4} \times 12$$

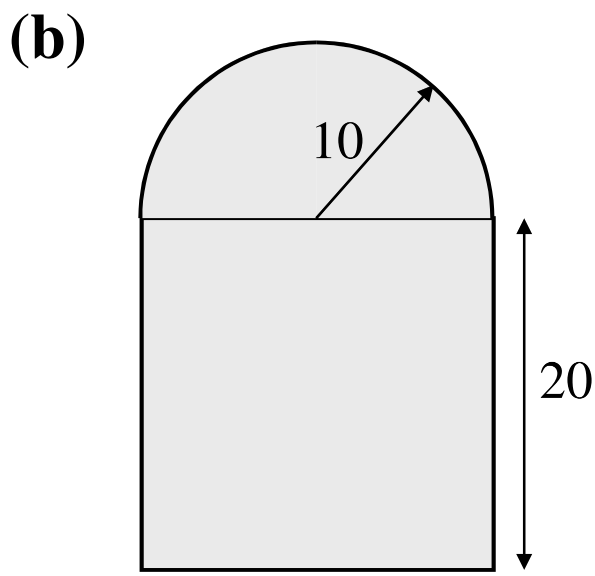
$$\therefore\ \bar{y} = 21{,}557.522 \div 1421.46 = \underline{\underline{15.1658\,\text{mm}}}$$

Similarly,

$$\bar{x} = 36{,}557.522 \div 1421.46 = \underline{\underline{25.7183\,\text{mm}}}$$

Exercise: Determine the centroid of the following areas:





All dimensions are in mm

Second Moments and the Product of Area

Second moments of an area *A* about *x* and *y* coordinates are defined as

$$I_{xx} = \int_A y^2 dA$$

$$I_{yy} = \int_A x^2 dA$$

I_{xx} and I_{yy} cannot be negative, in contrast to the first moments.

Similar to the concept of centroid, the entire area is assumed to be concentrated at (k_x, k_y) such that

$$A k_x^2 = I_{xx}$$

$$A k_y^2 = I_{yy}$$

Or

$$k_x = \sqrt{\frac{I_{xx}}{A}}$$

$$k_y = \sqrt{\frac{I_{yy}}{A}}$$

The distances k_x and k_y are called radii of gyration.

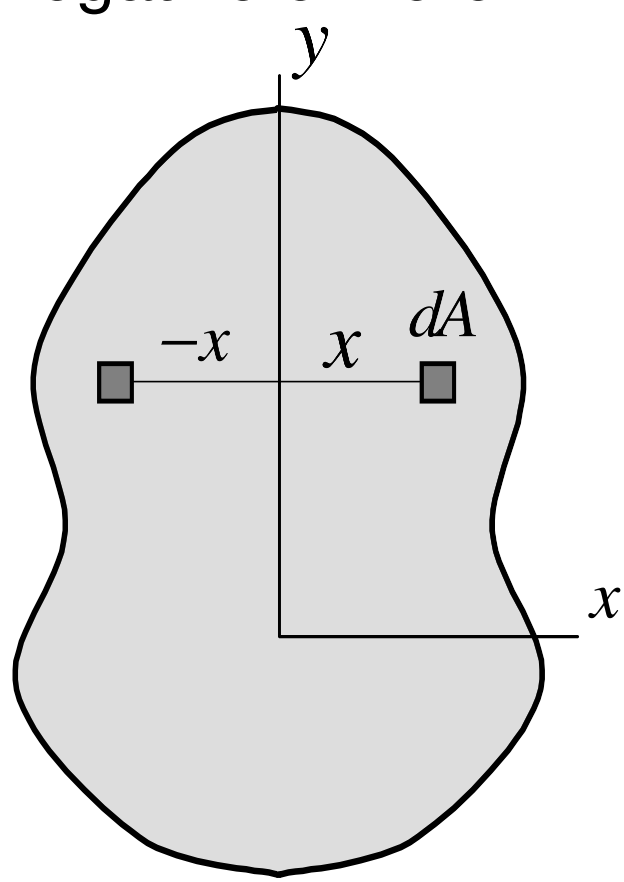
They depend on both the shape of the area and the position of the *x*, *y* axes (unlike centroid).

The product of area is defined as

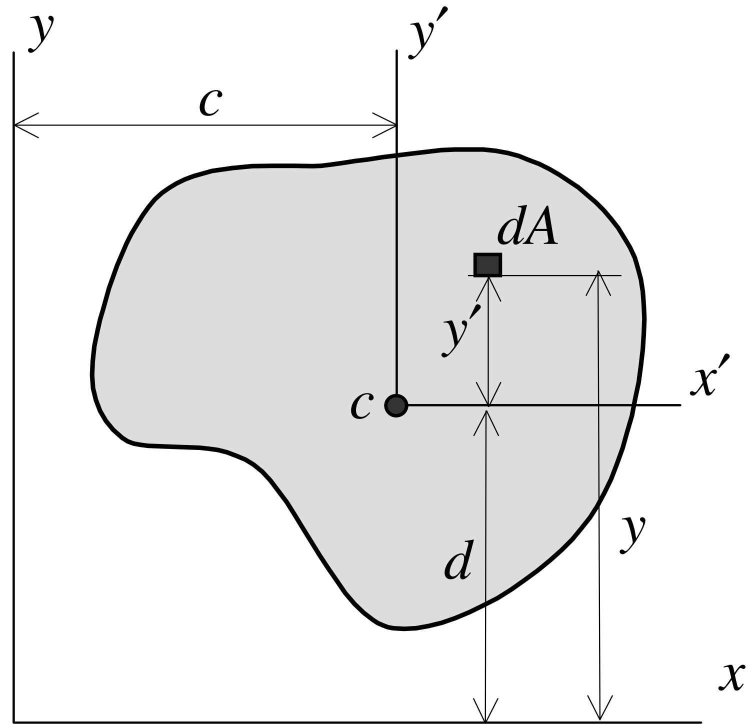
$$I_{xy} = \int_A xy\, dA$$

I_{xy} could be positive, negative or zero.

If the area has an axis of symmetry, the product of area for this axis and any axis orthogonal to this axis is zero.



Transfer Theorems



In the above figure, x' and y' are the centroidal axis.

$$I_{xx} = \int_A y^2 dA = \int_A (y'+d)^2 dA$$

$$= \int_A (y')^2 dA + 2d\int_A y' dA + A d^2$$

$$= I_{x'x'} + A d^2$$

I about any axis = I about any parallel axis through centroid + $A\,d^2$

$$I_{xy} = \int_A xy\,dA = \int_A (x'+c)(y'+d)\,dA$$

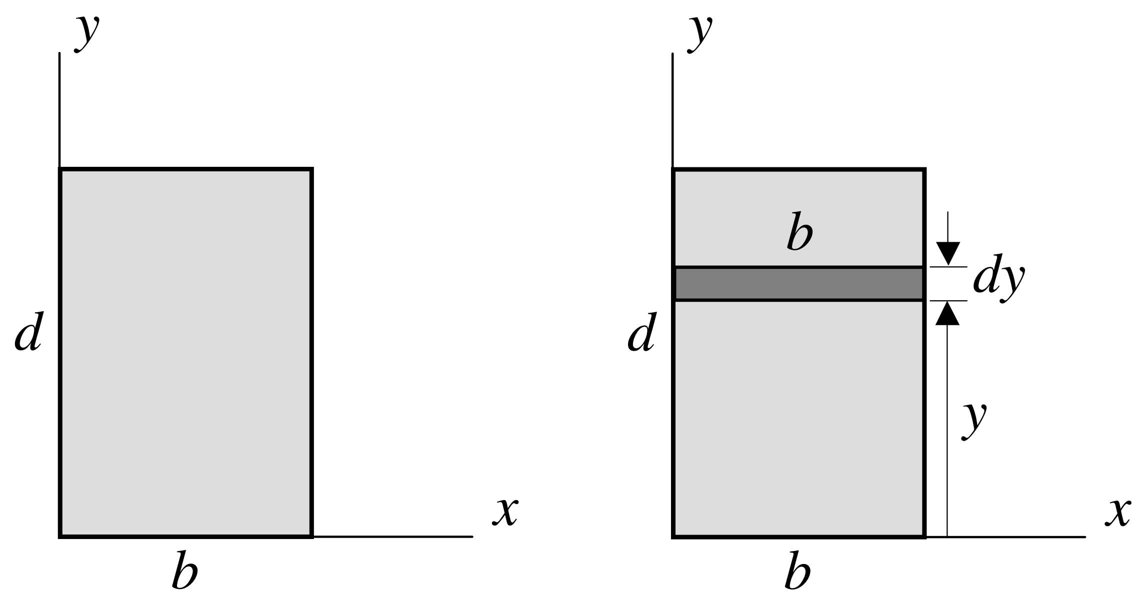
$$= \int_A x'\,y'\,dA + c\int_A y'\,dA + d\int_A x'\,dA + cd\int_A dA$$

$$= I_{x'y'} + c\,d\,A$$

I_{xy} about any axis = $I_{x'y'}$ about any parallel axis through centroid + $c\,d\,A$

Important Note: The distances c and d are measured from the x and y axes to the centroid.

Example: Find I_{xx}, I_{yy} and I_{xy} of a rectangle of size *b* and *d* about the *x* and *y* axes shown in figure.

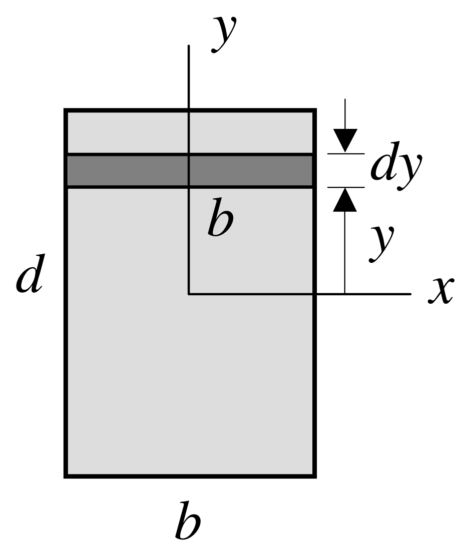


$$I_{xx} = \int_A y^2 dA = \int_0^d y^2\, b dy = \frac{bd^3}{3}$$

$$I_{yy} = \int_A x^2 dA = \int_0^b x^2\, d\, dx = \frac{db^3}{3}$$

$$I_{xy} = \int_A xy\, dA = \int_0^b \int_0^d xy\, dx\, dy = \frac{b^2 d^2}{4}$$

Example: Find I_{xx}, I_{yy} and I_{xy} of the rectangle of size *b* and *d* about the centroidal *x* and *y* axes shown in figure.



$$I_{xx} = \int_A y^2 dA = \int_{-d/2}^{d/2} y^2\, b dy = \frac{bd^3}{12}$$

Similarly, we get

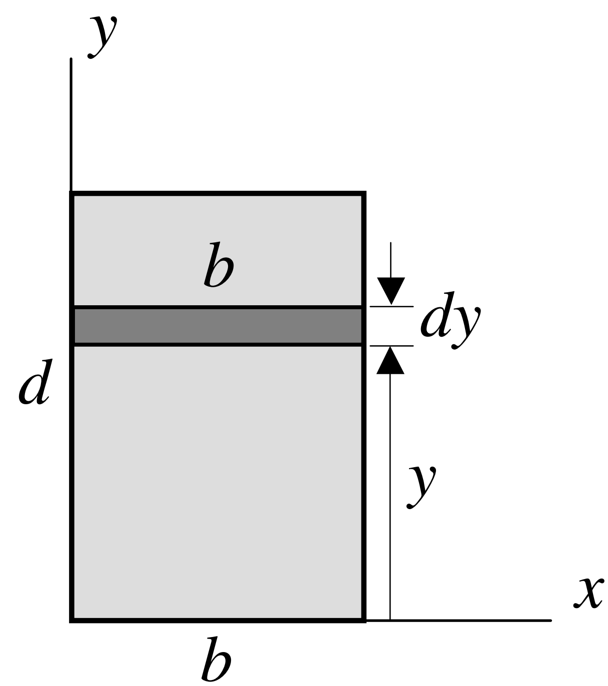
$$I_{yy} = \int_A x^2 dA = \int_{-b/2}^{b/2} x^2\, d\, dx = \frac{db^3}{12}$$

and

$$I_{xy} = \int_A xy\, dA = \int_{-b/2}^{b/2} \int_{-d/2}^{d/2} xy\, dx\, dy = 0$$

which is due to the lines of symmetry.

From the above example, we can see the validity of transfer theorems. Thus, we have the second moment I_{xx} about the base is

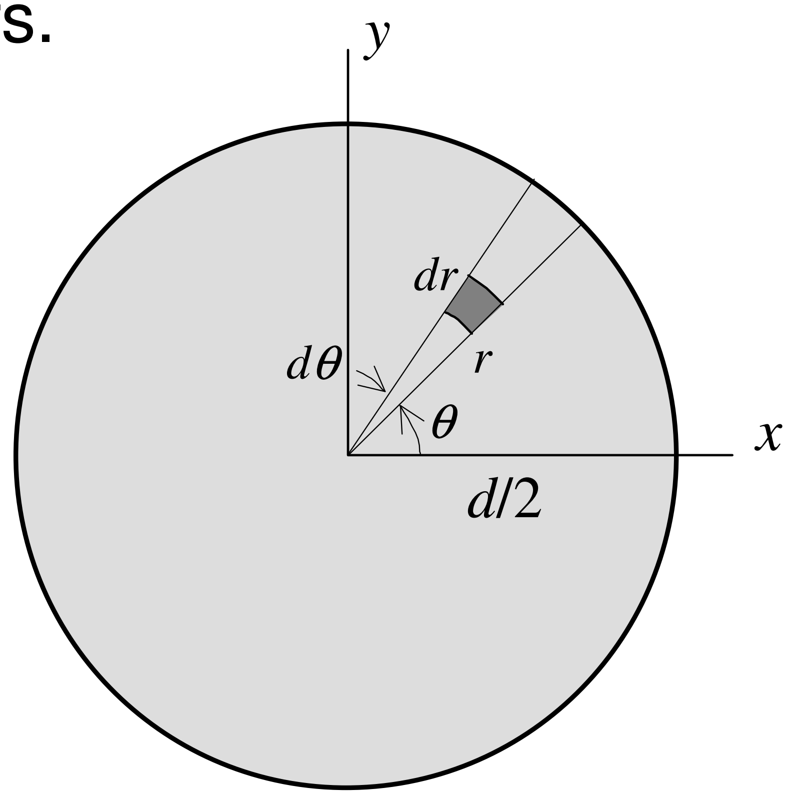


$$I_{xx} = \frac{bd^3}{12} + (bd)\left(\frac{d}{2}\right)^2 = \frac{bd^3}{12} + \frac{bd^3}{4} = \frac{bd^3}{3}$$

and

$$I_{xy} = \frac{b^2d^2}{16} + (bd)\left(\frac{b}{2}\right)\left(\frac{d}{2}\right) = 0 + \frac{b^2d^2}{4} = \frac{b^2d^2}{4}$$

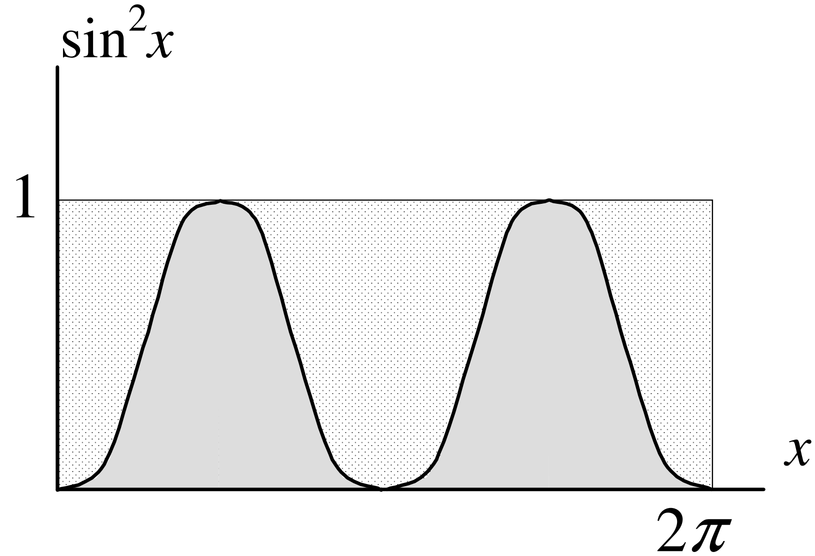
Example: Find I_{xx} of a circle about one of its diameters.



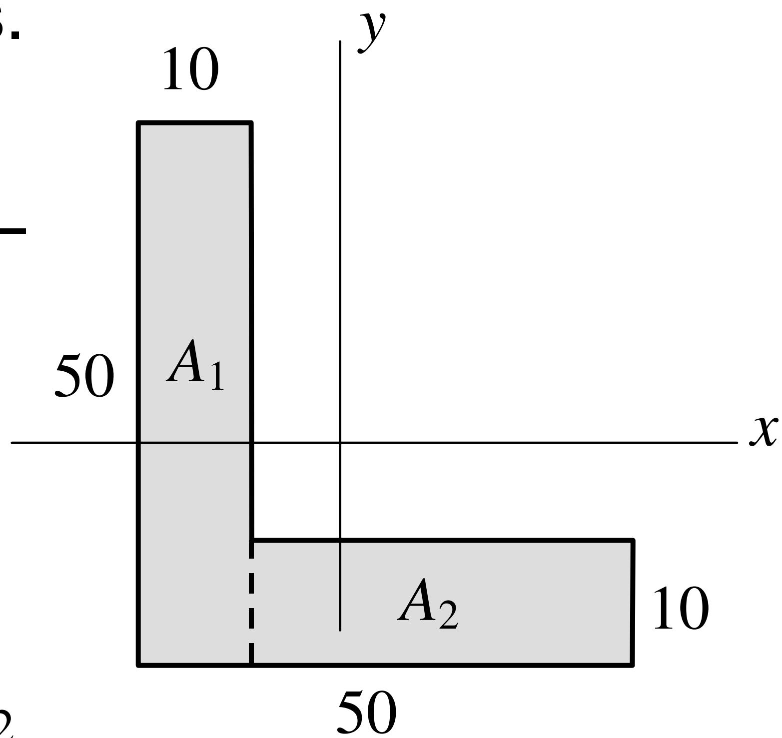
$I_{xy} = 0$ as there are an infinite number of lines of symmetry!

$$I_{xx} = \int_A y^2 dA = \int_0^{d/2} \int_0^{2\pi} (r \sin\theta)^2 (r\, dr\, d\theta)$$

$$= \frac{\pi d^4}{64} = I_{yy}$$



Example: Determine I_{xx}, I_{yy} and I_{xy} of the section shown in figure about its centroidal axes.



Dividing the area into two—A_1 and A_2—we get:

$A_1 = 500 \text{ mm}^2$
$A_2 = 400 \text{ mm}^2$

& $\underline{\underline{A = 900 \text{ mm}^2}}$

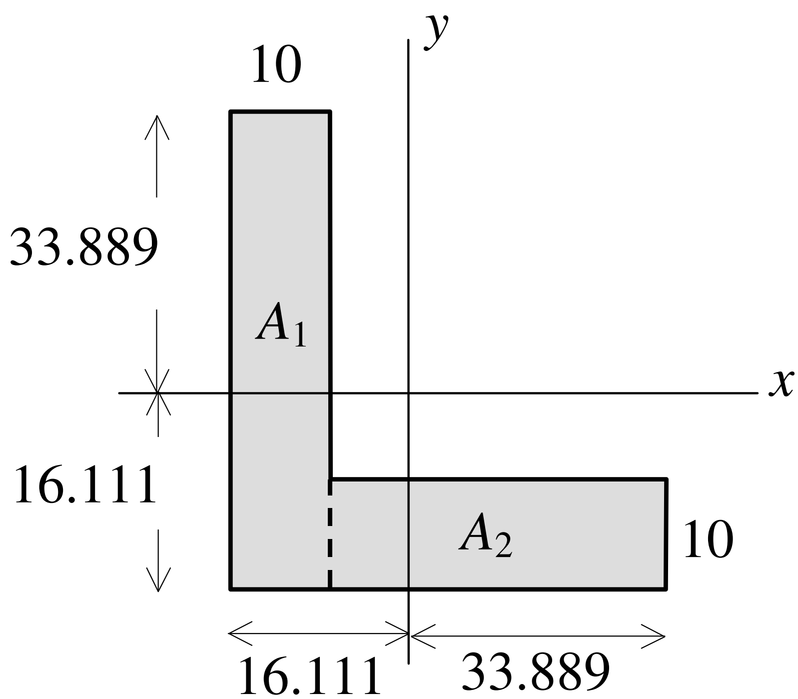
The centroidal distances x_c and y_c are obtained by taking moments of the area about the bottom and left side edges. Thus, we obtain

$$900 y_c = 400 \times 5 + 500 \times 25$$

$$\therefore \; \underline{\underline{y_c = 16.111 \text{ mm}}}$$

Due to symmetry, $x_c = y_c$

Redraw the figure, now marking the centroidal distances too. Thus



$$I_{xx} = \tfrac{1}{12}(10)(50)^3 + (500)(33.889 - 25)^2$$

$$+ \tfrac{1}{12}(40)(10)^3 + (400)(16.111 - 5)^2$$

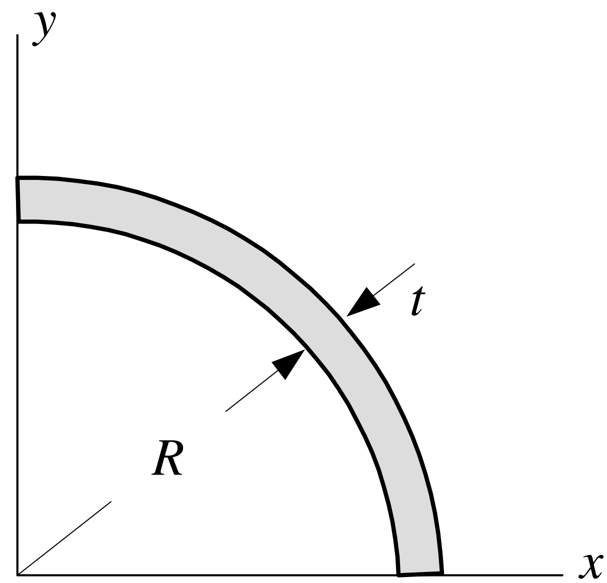
$$= 1.96389 \times 10^5 \,\text{mm}^4 = I_{yy}$$

$$I_{xy} = (-11.111)(8.889)(500)$$

$$+ (20 - 16.111)(-11.111)(400)$$

$$= -0.66667 \times 10^5 \,\text{mm}^4$$

Example: Determine I_{xx}, I_{yy} and I_{xy} of the section shown in figure about the *x* and *y* axes shown. Also determine the centroid and calculate the second moments and the product of the area about the centroidal axes. Given *t* is very small when compared to *R*.



$$I_{xx} = \int_0^{\pi/2} (R\sin\theta)^2 R\, d\theta\, t$$

$$= \frac{\pi R^3 t}{4} = I_{yy}$$

$$I_{xy} = \int_0^{\pi/2} (R\sin\theta)(R\cos\theta)\, R\, d\theta\, t = \frac{R^3 t}{2}$$

The remaining part is “homework”.

Coordinate Transformation—Concept of Cartesian Tensor

A. Scalar (Zeroth order tensor)

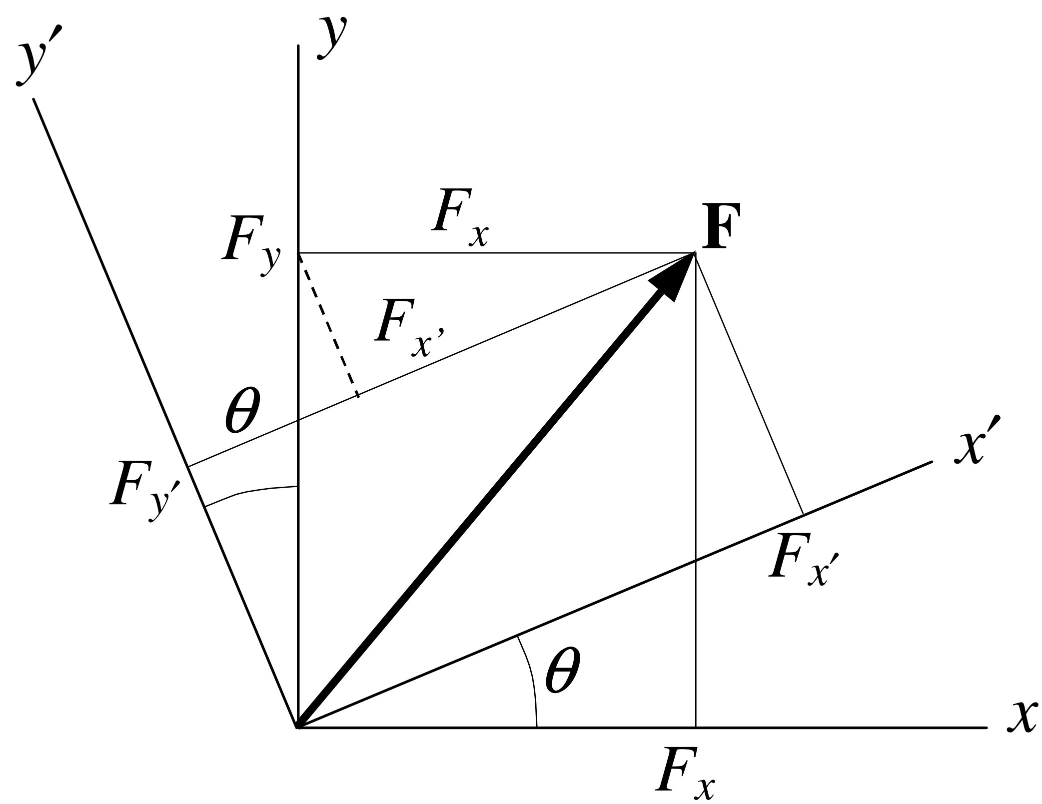
A scalar quantity, say the temperature T at a point, remains invariant when the coordinate axes are rotated.

B. Vector (First order tensor)

Next, let us consider a vector, say a force vector **F**.

Although the vector as such remains the same, its components keep varying as the coordinate axes are rotated.

The question is this: If F_x and F_y components of **F** are known with respect to x and y coordinates, determine the components $F_{x'}$ and $F_{y'}$ with respect to x' and y' coordinates, which are obtained by rotating x and y by an angle θ.



From the above, it is easy to verify that

$$F_{x'} = F_x \cos\theta + F_y \sin\theta$$

and

$$F_{y'} = -F_x \sin\theta + F_y \cos\theta$$

which can be written using matrix notation as

$$\begin{Bmatrix} F_{x'} \\ F_{y'} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

The above transformation can be written more concisely as

$$\{\mathbf{F}'\} = [R]\{\mathbf{F}\} \qquad \boxed{\mathbf{A}}$$

where

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is called the *rotation transformation matrix*. It is an orthogonal matrix (which means that $R^{-1} = R^T$ as $RR^T = I$).

Eq. [A] represents the transformation law for vectors (in other words, all the quantities that transform according to Eq. [A] are called vectors).

The position vector at a point also transforms according to the same law which can be written as

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

B. Second Order Tensor—Dyadic

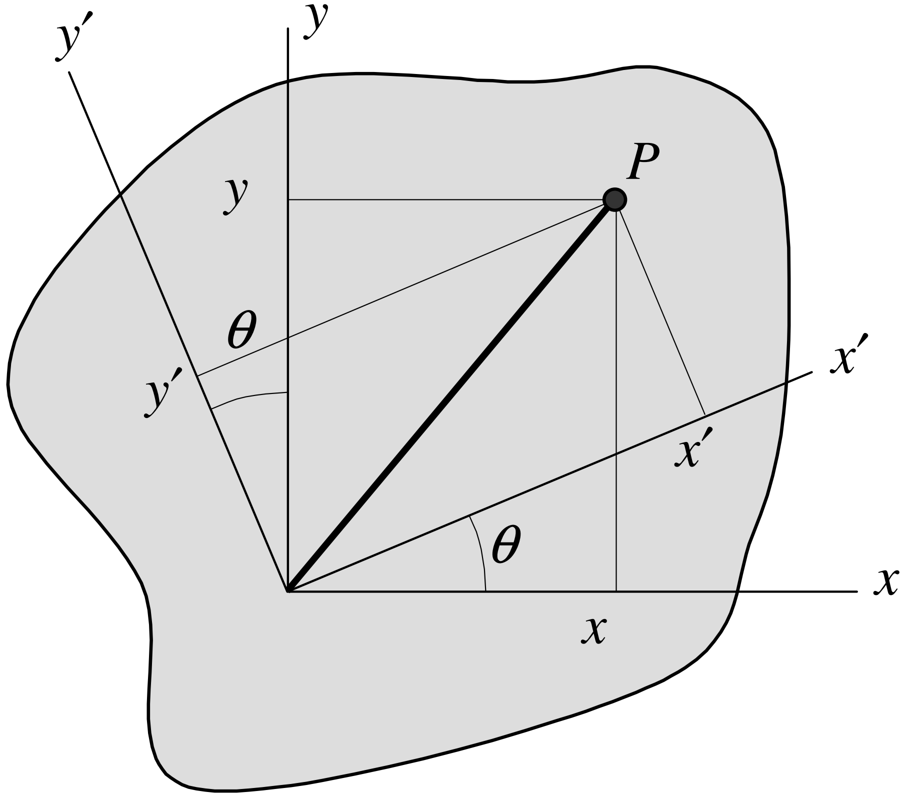
The second moment of area and the product of area put together for a given area at a point can be written as follows:

$$[I_x] = \begin{bmatrix} I_{xx} & -I_{xy} \\ -I_{xy} & I_{yy} \end{bmatrix}$$

which is a symmetric matrix. Now the question is this:

Given the above components of the second moment area tensor with respect to *x* and *y* coordinates, determine the components ($I_{x'x'}$, $I_{y'y'}$ and $I_{x'y'}$) with respect to *x'* and *y'* coordinates, which are obtained by rotating *x* and *y* by an angle θ.

We proceed as follows.



$$I_{x'x'} = \int_A (y')^2\, dA$$

$$= \int_A (-x\sin\theta + y\cos\theta)^2\, dA$$

$$= I_{yy}\sin^2\theta + I_{xx}\cos^2\theta - 2I_{xy}\sin\theta\cos\theta$$

which is written as

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$I_{y'y'}$ is obtained from the above by replacing θ by $\theta + \pi/2$. Thus, we obtain

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

Similarly, we obtain $I_{x'y'}$ as

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

The above three results can be represented using matrix notation as

$$\begin{bmatrix} I_{x'x'} & -I_{x'y'} \\ -I_{x'y'} & I_{y'y'} \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} \\ -I_{xy} & I_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Or

$$[I_{x'}] = [R][I_x][R]^T$$

Principal Axes

Thus, we have seen that as the angle θ changes, we start getting different components for the area tensor.

Now the question arises: Is there any value of θ at which $I_{x'x'}$ takes on a maximum (or minimum value)?

$I_{x'x'}$ is maximum when

$$\frac{\partial I_{x'x'}}{\partial \theta} = \frac{I_{xx} - I_{yy}}{2}(-2\sin 2\theta) - 2I_{xy}\cos 2\theta = 0$$

or

$$\tan 2\bar{\theta} = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

where $\bar{\theta}$ corresponds to an extreme value of $I_{x'x'}$.

There are two possible values of $\bar{\theta}$ which are $\pi/2$ radians apart.

The axes corresponding to these angles are called the *principal axes*.

The second moments of area about these axes are called the *principal moments of area*—one being the *major principal moment* and the other the *minor principal moment.*

Next, let us determine the product of area about the principal axes.

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} \sin 2\overline{\theta} + I_{xy} \cos 2\overline{\theta}$$

Dividing the above by cos $2\overline{\theta}$ we get

$$\frac{I_{x'y'}}{\cos 2\overline{\theta}} = \frac{I_{xx} - I_{yy}}{2} \tan 2\overline{\theta} + I_{xy} = 0$$

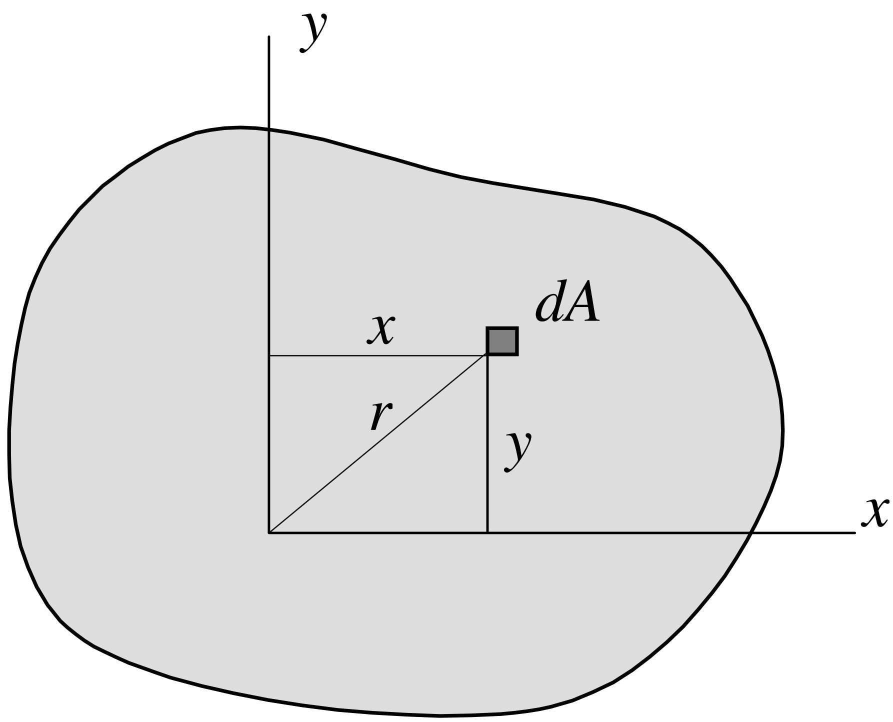
Thus, we see that the product of area is *zero* about the principal axes.

Moreover, we see that

$$I_{x'x'} + I_{y'y'} = I_{xx} + I_{yy}$$

is a constant.

Now, for any orthogonal set of axes we have



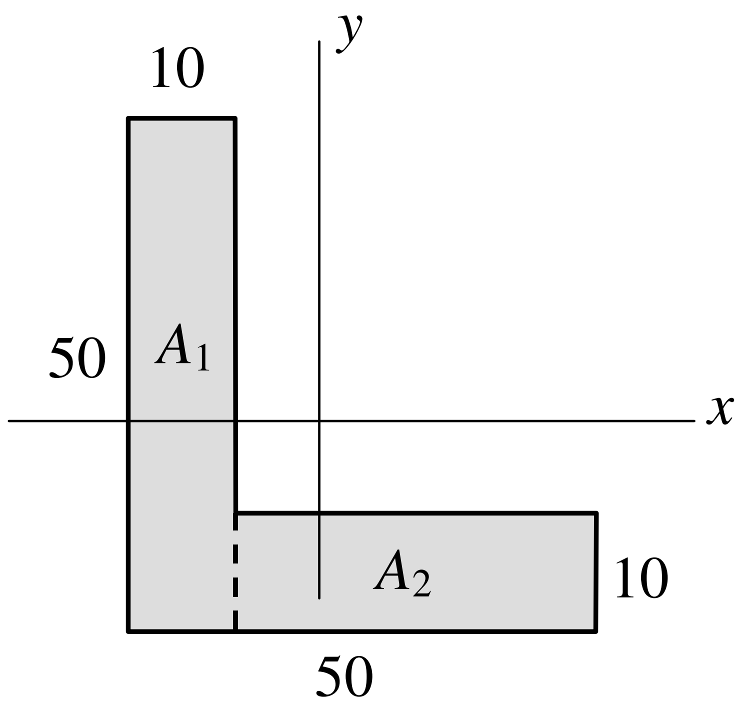
$$I_{xx} + I_{yy} = \int_A (y^2 + x^2) dA = \int_A r^2 \, dA$$

$I_{xx} + I_{yy}$ is called the *polar moment of area* denoted by J or I_P and is independent of the orientation of the axes.

Since $I_{xx} + I_{yy}$ = a constant, it is termed as an *invariant*. We can also show that $I_{xx} I_{yy} - I_{xy}^2$ is also an invariant under rotation of axes.

<u>Example</u>: Determine the principal moments of area of the plane area shown below about the centroidal axes.

We have seen earlier that the area, the centroidal distances and the moment area tensor are



$$A = 900\ \text{mm}^2$$

$$x_c = y_c = 16.111\ \text{mm}$$

$$I_{xx} = I_{yy} = 1.96389 \times 10^5\ \text{mm}^4$$

$$I_{xy} = -0.66667 \times 10^5\ \text{mm}^4$$

The principal axes are given by

$$\tan 2\overline{\theta} = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{2I_{xy}}{0} = \infty$$

or, $2\overline{\theta} = \pi/2$.

That is, $\overline{\theta}_1 = \pi/4$ and $\overline{\theta}_2 = 3\pi/4$.

The principal moments of area are:

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$= 1.96389\times10^5 + 0$$

$$-(-0.6667\times10^5)\sin 2(\pi/4)$$

$$= 2.6306\times10^5 \text{ mm}^4$$

and

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$= 1.96389\times10^5 - 0.6667\times10^5$$

$$= 1.2972\times10^5 \text{ mm}^4$$

It may be verified that

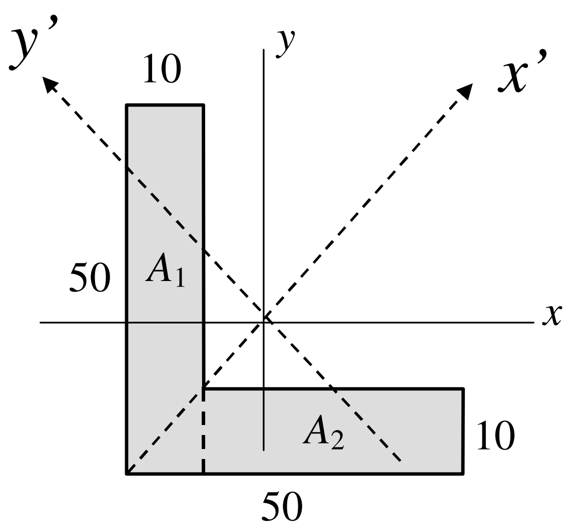
$$I_{x'x'} + I_{y'y'} = I_{xx} + I_{yy} = 3.9278 \times 10^5 \,\text{mm}^4$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} \sin 2\bar{\theta} + I_{xy} \cos 2\bar{\theta} = 0$$

$$\begin{bmatrix} I_{x'x'} & -I_{x'y'} \\ -I_{x'y'} & I_{y'y'} \end{bmatrix} =$$

$$\begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} \\ -I_{xy} & I_{yy} \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix}$$

$I_{xx} I_{yy} - I_{xy}^2$ can also be verified to remain unchanged during the transformation.



Principal Moments of Area as an Eigenvalue Problem

Find the direction cosines *l* and *m* of the principal axis (the axis which corresponds to an extreme value for $I_{x'x'}$.

Let $l = \cos\theta$ and $m = \sin\theta$

Then, the equation for $I_{x'x'}$ is given by

$$I_{x'x'} = I_{xx}\, l^2 + I_{yy}\, m^2 - 2I_{xy}\, l\, m$$

Let us maximise this with respect to the constraint that $l^2 + m^2 = 1$

This can be done using the Lagrange multiplier technique: Thus, maximise

$$F = I_{xx}\, l^2 + I_{yy}\, m^2 - 2I_{xy}\, l\, m - \lambda(l^2 + m^2 - 1)$$

which leads to

$$\frac{\partial F}{\partial l} = 2I_{xx}\, l - 2I_{xy}\, m - 2\lambda\, l = 0$$

That is $I_{xx}\, l - 2I_{xy}\ m = 2\lambda\, l$

Similarly, equating derivative of *F* with respect to *m* to zero and putting them together, we get

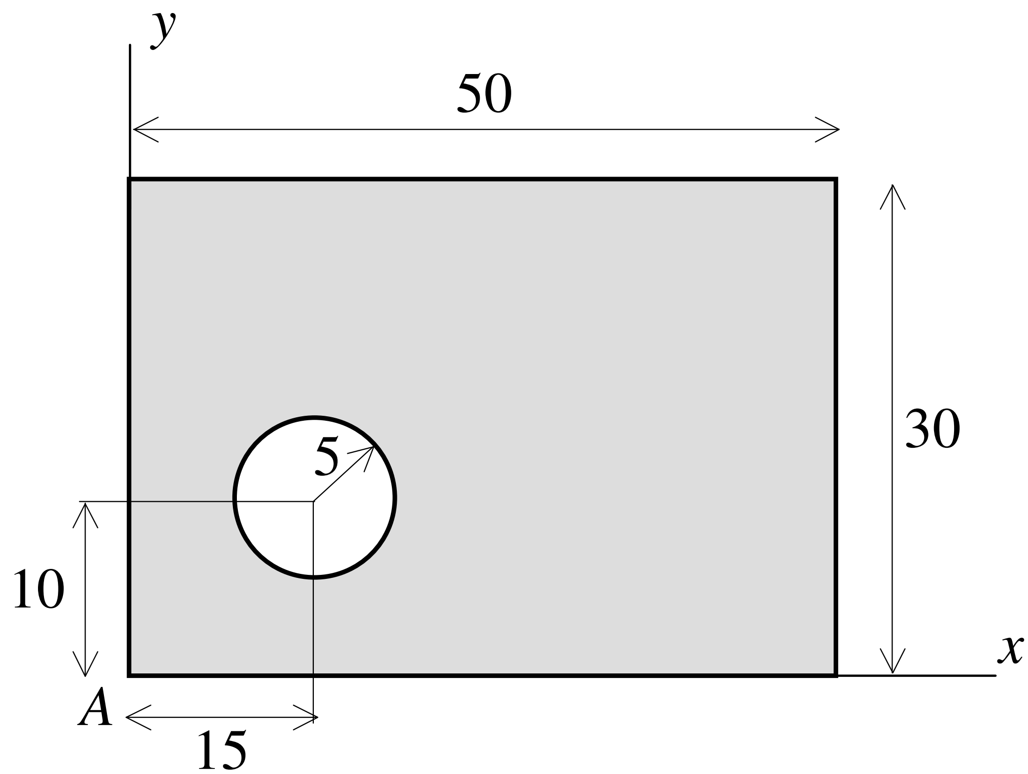
$$\begin{bmatrix} I_{xx} & -I_{xy} \\ -I_{xy} & I_{yy} \end{bmatrix} \begin{Bmatrix} l \\ m \end{Bmatrix} = \lambda \begin{Bmatrix} l \\ m \end{Bmatrix}$$

which is a matrix eigenvalue problem of the form $[A]\{X\} = \lambda\{X\}$

As the matrix $[I_x]$ is symmetric, the eigenvalues are always real (and not complex).

Hence, the principal moments of area are always real.

<u>Exercise</u>: Determine the principal axes and the principal moments of area of the following plane area at the point *A*:



$$I_{xx} = \frac{1}{12}(50)(30)^3 + (50\times 30)(15)^2$$

$$-\frac{\pi(10)^4}{64} - \frac{\pi(10)^2}{4}\times(10)^2 = \underline{\underline{0.441655\times 10^6}}\text{ mm}^4$$

$$I_{yy} = \frac{1}{12}(30)(50)^3 + (50\times 30)(25)^2$$

$$-\frac{\pi(10)^4}{64} - \frac{\pi(10)^2}{4}\times(15)^2 = \underline{\underline{1.23184\times 10^6}}\text{ mm}^4$$

$$I_{xy} = (50\times 30)(25)(15) - \frac{\pi(10)^2}{4}\times(15)(10)$$

$$= \underline{\underline{0.550719\times 10^6}}\text{ mm}^4$$

Thus the area tensor is given by:

$$[I_x] = \begin{bmatrix} 0.44166 & -0.55072 \\ -0.55072 & 1.23184 \end{bmatrix} \times 10^6$$

Solving the eigenvalue problem, we get

$$\begin{vmatrix} 0.44166-\lambda & -0.55072 \\ -0.55072 & 1.23184-\lambda \end{vmatrix} \times 10^6 = 0$$

which corresponds to the characteristic equation given by

$$\lambda^2 - I_1\lambda + I_2 = 0 \qquad \textbf{[A]}$$

where

$$I_1 = tr(I_x) = (0.44166 + 1.23184) \times 10^6$$

$$= 1.6735 \times 10^6 \text{ mm}^4$$

and

$$I_2 = \det(I_x) = (0.44166 \times 1.23184 - 0.55072^2) \times 10^{12}$$

$$= 0.240762 \times 10^6 \text{ mm}^4$$

The solution of equation [A] gives the eigenvalues.

$$\lambda_1 = 1.5145 \times 10^6 \text{ and } \lambda_2 = 0.158968 \times 10^6$$

Determine the principal axes and the principal moments of areas for the following sections at *A*.