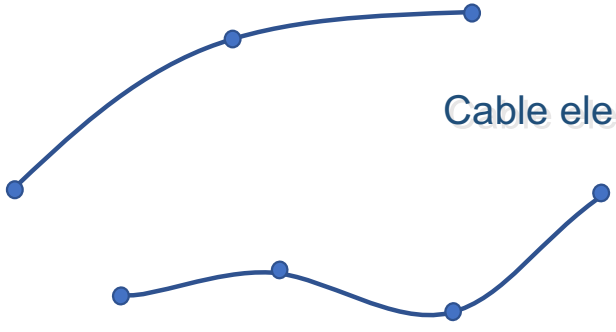
Isoparametric Formulation

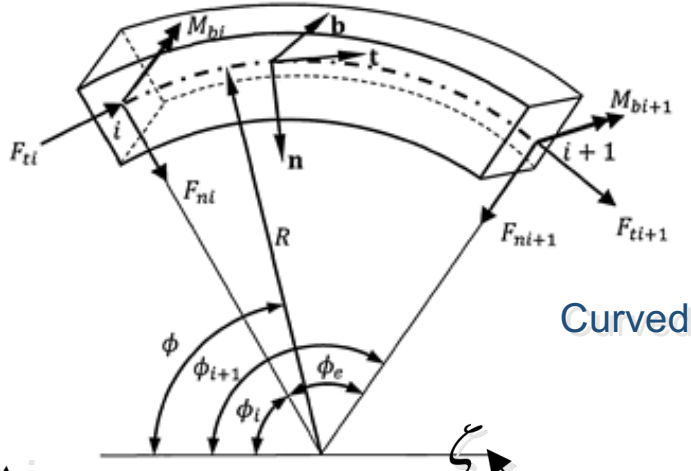
Introduction:

- Isoparametric formulation was publicly introduced in 1966 by Irons though it was invented first by Taig in 1958
- Leads to nonrectangular quadrilateral elements
- Makes use of special coordinate system ($\xi\eta\zeta$) to arrive at elements with odd shapes and curved sides
- Useful in modelling curved boundaries; in grading a mesh from coarse to fine
- Isoparametric formulation is useful in 1D: curved beams, cables
- In 2- and 3D elasticity, plate and shell problems, and nonstructural applications like fracture mechanics problems
- Relies on numerical integration, usually Gauss quadrature, for getting the element stiffness matrices

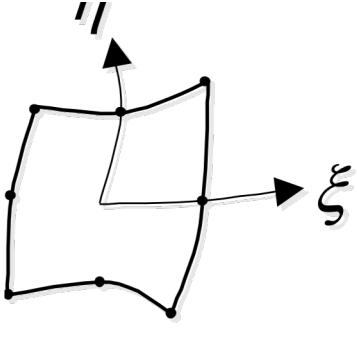
Sample Isoparametric Elements



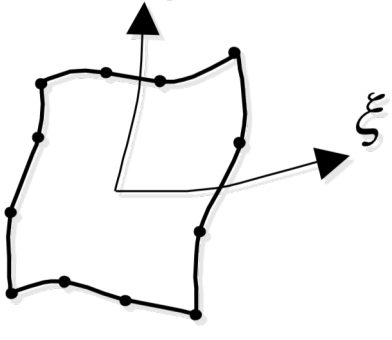
Cable elements



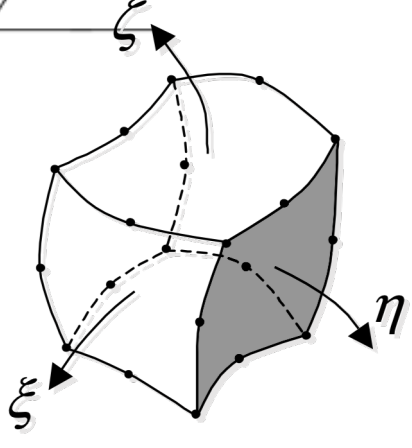
Curved beam element



Quadratic plane element



Cubic plane element



Quadratic hexahedral element

Concept

- The displacement field within the element is interpolated using the interpolation functions N_i

$$u = \sum N_i u_i ; v = \sum N_i v_i$$

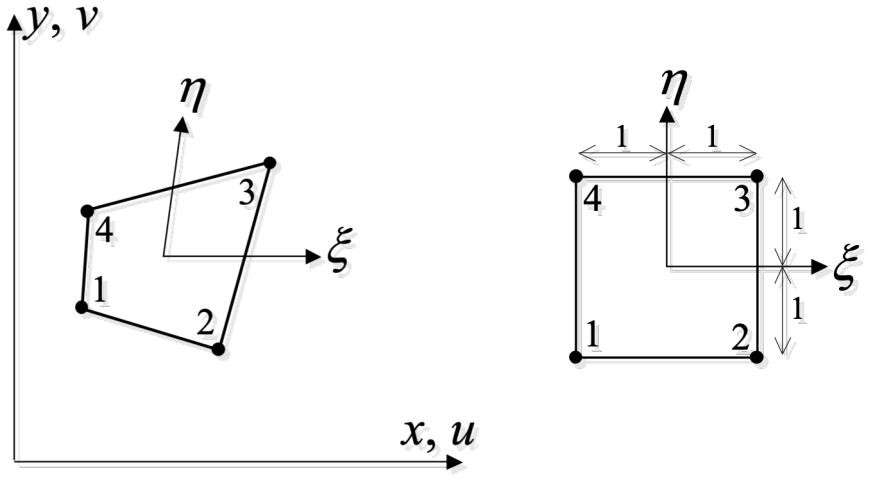
- The global coordinates of an internal point are interpolated using the interpolation functions $\bar{N}_i$

$$x = \sum \bar{N}_i x_i ; \; y = \sum \bar{N}_i y_i$$

- If $\bar{N}_i$ is of degree less than N_i subparametric
- If $\bar{N}_i$ is of degree greater than N_i superparametric
- If $\bar{N}_i$ is of same degree as that of N_i *isoparametric* formulation
- However, it can be shown that only the isoparametric formulation is mathematically correct

Example: Plane linear isoparametric element

- Consider the four-noded isoparametric element shown in figure:



- The displacement field within the element is given by:

$$u = \sum N_i u_i \; ; \quad v = \sum N_i v_i$$

- While the global coordinates of an internal point are:

$$x = \sum N_i x_i \qquad y = \sum N_i y_i$$

- The Lagrangian interpolation can be used. Thus, we get

$$N_1 = (1 - \xi)(1 - \eta) \qquad N_2 = (1 + \xi)(1 - \eta)$$

$$N_3 = (1 + \xi)(1 + \eta) \qquad N_4 = (1 - \xi)(1 + \eta)$$

- Since N_i are functions of ξ and η, we need to make use of the chain rule to obtain the derivatives:

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

which in matrix notation is

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

or

$$\begin{Bmatrix} f_{,\xi} \\ f_{,\eta} \end{Bmatrix} = [J] \begin{Bmatrix} f_{,x} \\ f_{,y} \end{Bmatrix}$$

where J is called the Jacobian matrix

- Its inverse relation is:

$$\begin{Bmatrix} f_{,x} \\ f_{,y} \end{Bmatrix} = [\Gamma] \begin{Bmatrix} f_{,\xi} \\ f_{,\eta} \end{Bmatrix}$$

where $\Gamma = J^{-1}$.

The Jacobian matrix is obtained from the following:

$$x = \sum N_i x_i \qquad J = \begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix}$$
$$y = \sum N_i y_i$$

Hence,

$$J_{11} = \sum N_{i,\xi} x_i = N_{1,\xi} x_1 + N_{2,\xi} x_2 + N_{3,\xi} x_3 + N_{4,\xi} x_4$$

$$J_{12} = \sum N_{i,\xi} y_i = N_{1,\xi} y_1 + N_{2,\xi} y_2 + N_{3,\xi} y_3 + N_{4,\xi} y_4$$

$$J_{21} = \sum N_{i,\eta} x_i = N_{1,\eta} x_1 + N_{2,\eta} x_2 + N_{3,\eta} x_3 + N_{4,\eta} x_4$$

$$J_{22} = \sum N_{i,\eta} y_i = N_{1,\eta} y_1 + N_{2,\eta} y_2 + N_{3,\eta} y_3 + N_{4,\eta} y_4$$

- Now, we need to determine the strain-displacement matrix $[B]$:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{[A]} \begin{Bmatrix} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{Bmatrix}$$

$$\begin{Bmatrix} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{Bmatrix} = \underbrace{\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix}}_{[\Gamma]} \begin{Bmatrix} u_{,\xi} \\ u_{,\eta} \\ v_{,\xi} \\ v_{,\eta} \end{Bmatrix}$$

$$\begin{Bmatrix} u_{,\xi} \\ u_{,\eta} \\ v_{,\xi} \\ v_{,\eta} \end{Bmatrix} = \begin{bmatrix} N_{1,\xi} & 0 & N_{2,\xi} & 0 & \dots \\ N_{1,\eta} & 0 & N_{2,\eta} & 0 & \dots \\ 0 & N_{1,\eta} & 0 & N_{2,\eta} & \dots \\ 0 & N_{2,\eta} & 0 & N_{2,\eta} & \dots \end{bmatrix} \{u^e\}$$

$[D_N]$

where $\{u^e\} = [u_1 \; v_1 \; u_2 \; v_2 \; \dots \; u_4 \; v_4]^{\mathrm{T}}$

For the present case of 4-noded isoparametric quadrilateral element, $[D_N]$ is given by:

$$[D_N] = \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}$$

- Matrix $[B]$ is obtained from the above three matrices as

$$[B] = [A]\,[\Gamma]\,[D_N]$$

- Matrix $[B]$ is obtained from the above three matrices as $[B] = [A]\,[\Gamma]\,[D_N]$
- The element stiffness matrix is given by:

$$[k^e]_{8\times 8} = \iint [B]^T_{8\times 3}\,[D]_{3\times 3}\,[B]_{3\times 8}\,t\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\,t\,J\,d\xi\,d\eta$$

where *t* is the thickness of the element

$$\{r^e\} = \int_{-1}^{1}\int_{-1}^{1} (B^T D\boldsymbol{\varepsilon}_0 - B^T\boldsymbol{\sigma}_0 + N^T\mathbf{b})t|J|\,d\xi d\eta$$

- The above integral, in general, is not amenable to closed form integration
- Hence numerical integration is used to evaluate the integral
- Gauss quadrature is almost always used for this purpose.

Gauss Quadrature

- "Quadrature" is same as numerical integration
- There are several quadrature rules; for example, Newton-Cotes rules such as the *Trapezoidal rule* or the *Simpson's rule*

One-Dimensional Integration:

- An integral with arbitrary limits can be transformed to one with limits from –1 to 1. With $f = f(x)$, the substitution

$$x = \tfrac{1}{2}(1-\xi)\,x_1 + \tfrac{1}{2}(1+\xi)\,x_2$$

can be used to transform

$$I = \int_{x_1}^{x_2} f\,dx \quad \text{to} \quad I = \int_{-1}^{+1} \phi\,d\xi$$

- Thus the integrand changes from $f = f(x)$ to $\phi = \phi(\xi)$, where ϕ incorporates the Jacobian of the transformation, $J = dx/d\xi = ½(x_1 - x_2)$ also
- The Gauss quadrature formula for the transformed integral with limits between –1 and +1 is given by:

$$I = \int_{-1}^{+1} \phi \, d\xi = W_1\phi_1 + W_2\phi_2 + \cdots + W_n\phi_n$$

- In the above $\phi_i = \phi(\xi_i)$, and W_i are the weights and ξ_i are the sampling points. These are well tabulated
- Gauss obtained the weights and sampling points such that the greatest accuracy is achieved for a given n
- A polynomial of degree $2n - 1$ is integrated exactly by n-point Gauss quadrature

Two- and Three-Dimensional Integration:

- Multidimensional Gauss rules are formed by successive application of the one-dimensional Gauss rules. Thus, in two-dimensional case:

$$I = \int_{-1}^{+1}\int_{-1}^{+1} \phi(\xi,\eta)\, d\xi\, d\eta = \sum_i \sum_j W_i W_j \phi(\xi_i, \eta_j)$$

- And, in three-dimensional case:

$$I = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \phi(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta$$

$$= \sum_i \sum_j \sum_k W_i W_j W_k \phi(\xi_i, \eta_j, \zeta_k)$$

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