Statically Indeterminate Structures

Force Method of Analysis of Indeterminate Structures

- Any structure whose reactions or internal stress resultants cannot be determined using the equations of static equilibrium alone is *statically indeterminate*
- Additional equations called *compatibility conditions* or *consistency conditions* are necessary
- A statically indeterminate structure is also termed as a *redundant structure*
- *Degree of Static Indeterminacy* (DSI) is the number of unknown reactions in excess of the number of equations of equilibrium

A. Continuous Beams

A statically indeterminate structure can be made *determinate* by identifying a number of *redundant reactions* (or simply *redundants*) equal in number to the DSI such that the resulting determinate structure is *stable*. Once the redundants are removed, the structure becomes a statically determinate one (known as the *primary* or *basic structure*).

Example 1: A propped cantilever with loads is shown in Fig. 1. DSI = 1 (= 4 – 3, the number of unknown reactions *minu*s the number of equations of equilibrium). The free body diagram is shown in Fig. 2.

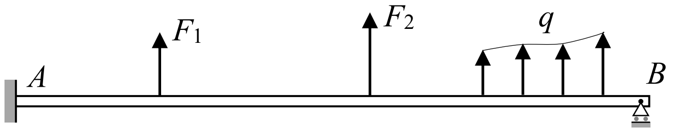

Figure 1

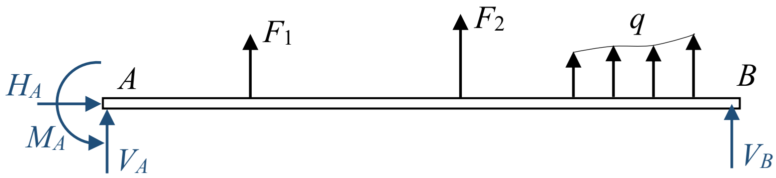

Figure 2

In the above, either V_B can be chosen as the only redundant (then the primary structure is a cantilever) or M_A can be chosen as the redundant (the primary structure then is a simply supported beam).

Example 2: A continuous beam is loaded as shown in Fig. 3 with DSI = 3 (= 6 – 3). The free body diagram is shown in Fig. 4.

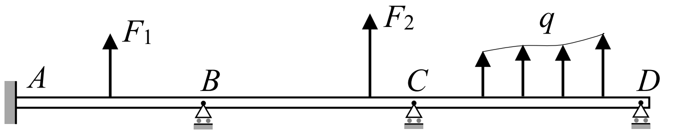

Figure 3

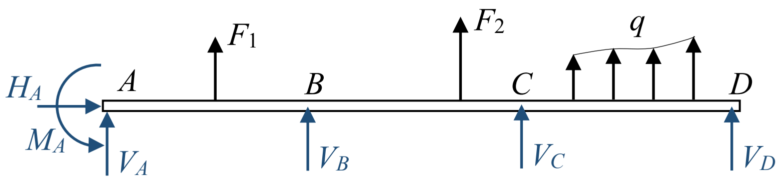

Figure 4

In the above, we may choose V_B, V_C and V_D as the redundants. Then the *primary* beam is a cantilever. An alternative is to consider M_A, V_B and V_C as the redundants. Then we end up with a simply-supported beam as the primary structure. The former choice with redundants is as shown in Fig. 5.

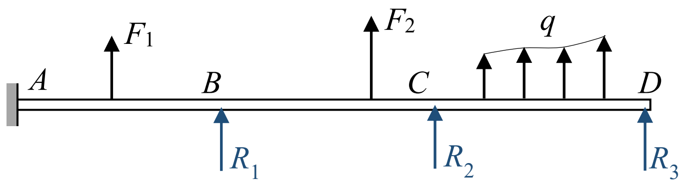

Figure 5

The redundants are R_1, R_2 and R_3 (the unknown reactions in excess of the number of equations of equilibrium). The redundants are determined using the compatibility conditions as described below. Let the deflections of the supports B, C and D be Δ_1, Δ_2 and Δ_3 respectively (positive along R_i). The determinate structure shown above is analysed by first removing all the redundants. Let Δ_{10}, Δ_{20} and Δ_{30} represent the deflections of the *primary structure* due to the applied loads as shown in Fig. 6.

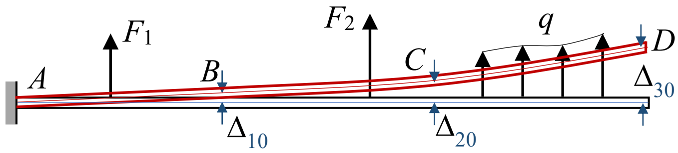

Figure 6

Then remove the external loads and determine the *flexibility influence coefficients* (f_{ij} denoting the deflection at the i^{th} redundant along R_i due to a unit load applied at j along R_j) by applying a unit load corresponding to each redundant one at a time. Fig. 7 depicts the flexibility coefficients when $R_1 = 1$.

Figure 7

Then, the compatibility condition at i^{th} redundant point is:

$$\Delta_{i0} + \sum_{j=1}^{n} f_{ij} R_j = \Delta_i, \text{ for } i = 1 \text{ to } n \text{ (here, } n = 3) \qquad (1.1)$$

Putting all such equations together, we get:

$$\begin{Bmatrix} \Delta_{10} \\ \Delta_{20} \\ \vdots \\ \Delta_{n0} \end{Bmatrix} + \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{Bmatrix}, \qquad (1.2)$$

which can be written concisely as $\mathbf{\Delta}_0 + [f]\,\mathbf{R} = \mathbf{\Delta}$, where $[f]$ denotes the flexibility matrix (a square matrix of size $n \times n$. The following exercises demonstrate the application of the method.

Exercise 1: The continuous beam in Fig. 8 is loaded as shown. Its supports B and C settle down by 4 mm and 7 mm respectively. The flexural stiffness of the beam is $EI = 1.2 \times 10^5$ kN m^2. Determine the support reactions and draw the shear force and bending moment diagrams. Also find the deflection under the 60 kN load.

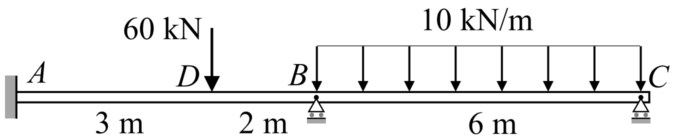

Figure 8

For the above beam, DSI = 2 with R_1 and R_2 as the redundants at B and C. The *primary structure* is shown in Fig. 9 below with the three separate loading cases: (a) due to the applied load, (b) due to $R_1 = 1$ and no R_2, and (c) due to $R_2 = 1$ and no R_1. Thus:

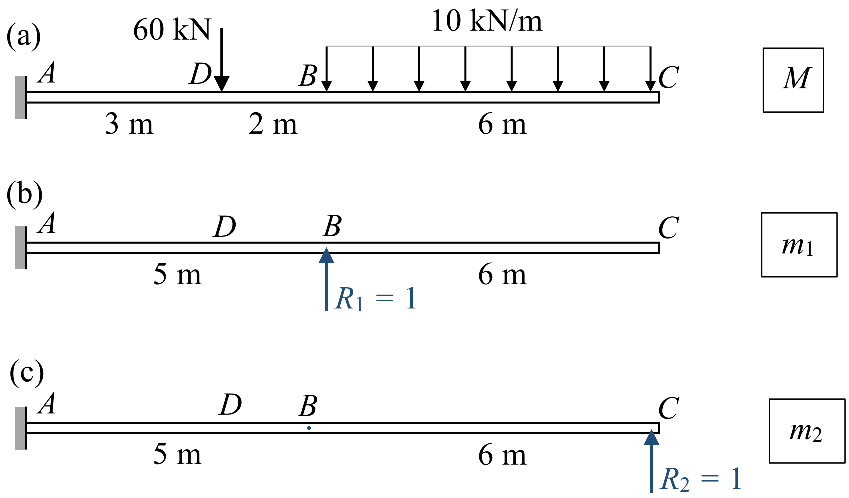

Figure 9

The deflections Δ_{i0} and f_{ij} are computed as shown in the table and the following lines.

Member	Origin	Limits (m)	M	m_1	m_2
CB	C	$0-6$	$-10\,x^2/2$	0	x
BD	B	$0-2$	$-60\,(x+3)$	x	$x+6$
DA	D	$0-3$	$-60\,(x+5)-60\,x$	$x+2$	$x+8$

$\Delta_{10} = \int_L \frac{Mm_1dx}{EI} = -\frac{5830}{EI}$; $\Delta_{20} = \int_L \frac{Mm_2dx}{EI} = -\frac{18970}{EI}$;

$f_{11} = \int_L \frac{m_1^2dx}{EI} = \frac{125}{3EI}$; $f_{12} = f_{21} = \int_L \frac{m_1m_2dx}{EI} = \frac{350}{3EI}$; $f_{22} = \int_L \frac{m_2^2dx}{EI} = \frac{1331}{3EI}$

The two compatibility equations for this beam are: $\Delta_{10} + f_{11}\,R_1 + f_{12}\,R_2 = \Delta_1$ and $\Delta_{20} + f_{21}\,R_1 + f_{22}\,R_2 = \Delta_2$
That is:

$$-\frac{5830}{EI} + \frac{125}{3EI}R_1 + \frac{350}{3EI}R_2 = -4\times10^{-3}, \text{ and}$$

$$-\frac{18970}{EI} + \frac{350}{3EI}R_1 + \frac{1331}{3EI}R_2 = -7\times10^{-3}.$$

Solving the above two algebraic equations simultaneously, we get $R_1 = 53.015$ kN and $R_2 = 26.923$ kN.

The end moments and support reactions then are:

$M_C = \underline{0 \text{ kN}}$; $M_B = R_2 \times 6 - 10\times6^2/2 = \underline{-18.462 \text{ kNm}}$;

$M_A = R_2 \times 11 + R_1 \times 5 - 10\times6\times(3+5) - 60\times3 = \underline{-98.772 \text{ kNm}}$;

and $V_A = 60 + 6\times10 - R_1 - R_2 = \underline{40.062 \text{ kN}}$

Once all the support reactions are known for the beam, it should not be difficult to plot the shear force and bending moment diagrams. The deflection under the 60 kN load can be determined using, for example, the unit load method. These are left as home-work exercises.

B. Indeterminate Frames

The equation that we derived for the continuous beam is equally valid for building frames also as given in Eq. (1.1). This is reproduced for convenience:

$$\Delta_{i0} + \sum_{j=1}^{n} f_{ij}R_j = \Delta_i, \text{ for } i = 1 \text{ to } n$$

where Δ_{i0} is the displacement in the primary structure (obtained by removing DSI number of redundants) corresponding to the redundant R_i due to the applied loading, f_{ij} are the flexibility coefficients and Δ_i is the final displacement of the indeterminate structure at i along R_i (which is normally zero except when the support settles down or moves by a known amount).

Exercise 2: Analyse, find all the support reactions and draw the BMD of the plane frame shown in Fig. 10.

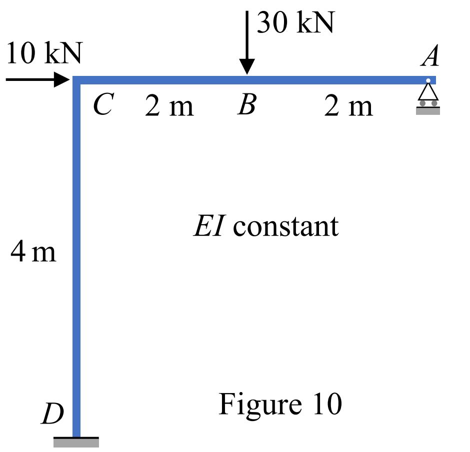

Figure 10

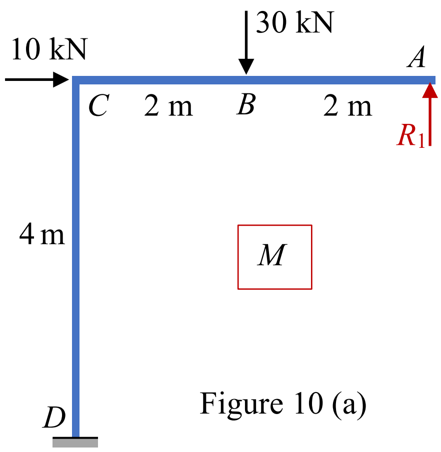

Figure 10 (a)

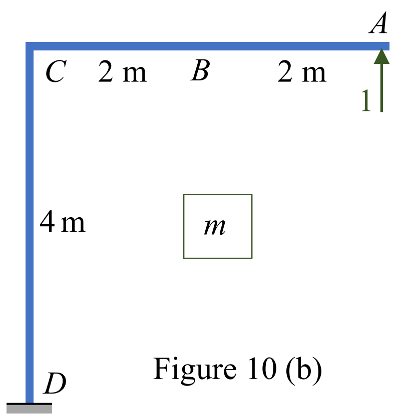

Figure 10 (b)

Here DSI = 1. Let us choose V_A (R_1) as the redundant as shown in Fig. 10 (a). Then we can tabulate M and m as shown below. The calculations shown beneath the table are self-explanatory.

Portion	Origin	Limits (m)	M	m
AB	A	$0-2$	0	$-x$
BC	B	$0-2$	$30x$	$-(2+x)$
CD	C	$0-4$	$60+10x$	-4

$$\Delta_{10} = \int_L \frac{Mm\,dx}{EI} = -\int_0^2 \frac{30x(2+x)dx}{EI} - \int_0^4 \frac{4(60+10x)dx}{EI} = -\frac{1480}{EI}$$

$f_{11} = \int_L \frac{m^2 dx}{EI} = \int_0^4 \frac{x^2 dx}{EI} + \int_0^4 \frac{16dx}{EI} = \frac{256}{3EI}$; We have $\Delta_{10} + f_{11}\,R_1 = \Delta_1 = 0$. Hence we get $R_1 = \frac{555}{3} = \underline{17.344 \text{ kN}}$

Once R_1 is known, we can use equations of equilibrium to determine the remaining support reactions. Then drawing the bending moment diagram is left as a homework.

Exercise 3: Analyse the portal frame shown in Fig. 11 and determine all the support reactions. Draw the BMD.

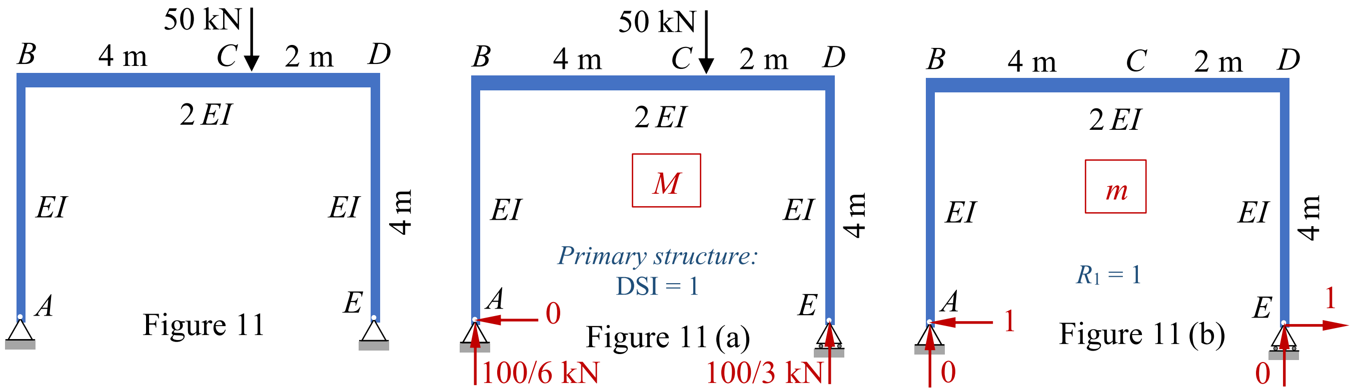

Figure 11   Figure 11 (a)   Figure 11 (b)

DSI = 1. Choose H_E (R_1) as the redundant. The support reactions due to the applied loading and due to $R_1 = 1$ are shown above in Figs. 11 (a) and (b). The bending moments M and m can then be tabulated as shown below. The deflection calculation steps are given beneath the table.

Portion	Origin	EI	Limits (m)	M	m
AB	A	EI	$0-4$	0	x
BC	B	$2\,EI$	$0-4$	$100x/6$	4
CD	D	$2\,EI$	$0-2$	$100x/3$	4
DE	E	EI	$0-4$	0	x

$$\Delta_{10} = \int_L \frac{Mm\,dx}{EI} = 0 + \int_0^4 \frac{400x\,dx}{6\times 2EI} + \int_0^4 \frac{400x\,dx}{3\times 2EI} = \frac{400}{EI}$$

$$f_{11} = \int_L \frac{m^2 dx}{EI} = 2\int_0^4 \frac{x^2 dx}{EI} + \int_0^4 \frac{4^2 dx}{2EI} + \int_0^2 \frac{4^2 dx}{2EI} = \frac{272}{3} = 90.667$$

$\Delta_{10} + f_{11}\,R_1 = \Delta_1 = 0;\ \therefore\ R_1 = \underline{-4.412 \text{ kN}}$

Once the redundant is found, all the support reactions can be calculated and the bending moment drawn. This part is left as an easy homework.

Exercise 4: Analyse the frame shown in Fig. 12 and determine all the support reactions. EI is constant.

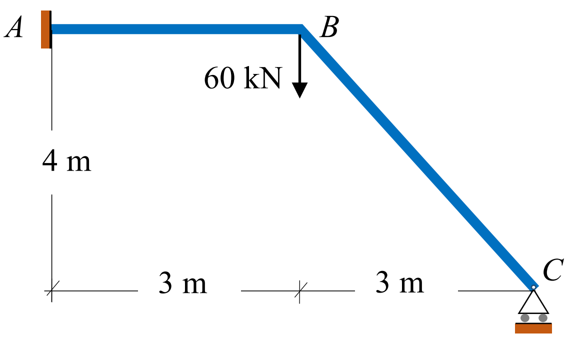

Figure 12

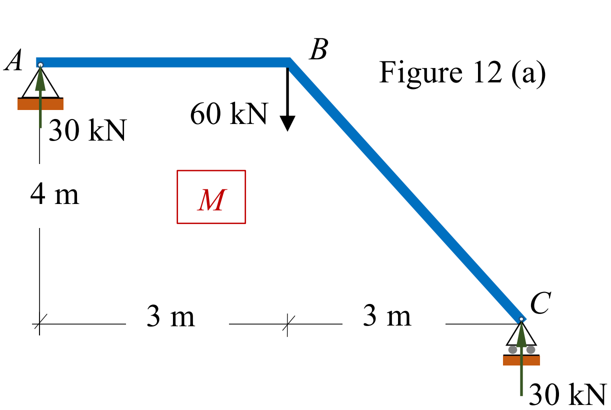

Figure 12 (a)

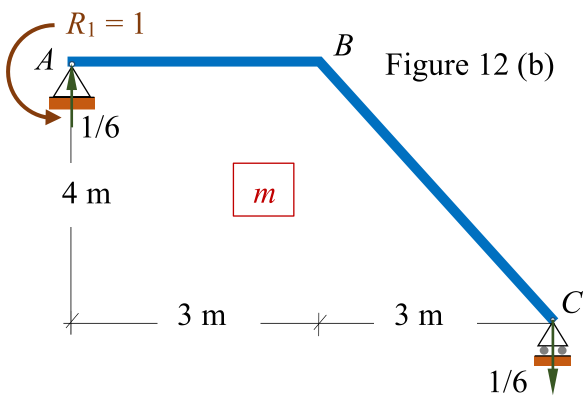

Figure 12 (b)

DSI = 1. Let us choose M_A as the redundant R_1. The primary structure with the applied load (Fig. 12a) and with $R_1 = 1$ (Fig. 12b) are shown. The table below shows values of M and m due to these two load cases. The deflection calculations follow the tabulation.

Portion	Origin	Limits (m)	M	m
AB	A	0 – 3	$30x$	$1/6\,x - 1$
BC	C	0 – 5	$30\,(0.6)x$	$-1/6\,(0.6)\,x$

$$\Delta_{10} = \int_L \frac{Mm\,dx}{EI} = \int_0^3 \frac{30x\,(x/6-1)dx}{EI} + \int_0^5 \frac{30(0.6)x\,(-0.1x)dx}{EI} = -\frac{165}{EI}$$

$$f_{11} = \int_L \frac{m^2dx}{EI} = \int_0^3 \frac{(x/6-1)^2dx}{EI} + \int_0^5 \frac{(-0.1x)^2dx}{EI} = \frac{13}{6EI}$$

$\Delta_{10} + f_{11}\,R_1 = \Delta_1 = 0$; $\therefore$ $R_1 = \underline{76.154 \text{ kN m}}$

Once R_1 (= M_A) is found, the remaining support reactions are calculated using the equations of equilibrium. They are shown in Fig. 12 (c) below:

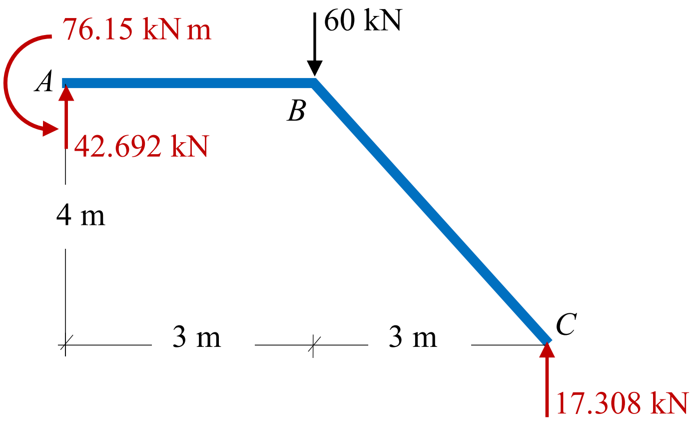

Figure 12 (c)

C. Indeterminate Trusses

A plane truss can be statically determinate or indeterminate. A statically indeterminate truss can be internally indeterminate, externally indeterminate, or both internally and externally indeterminate. The following figures are examples of these cases:

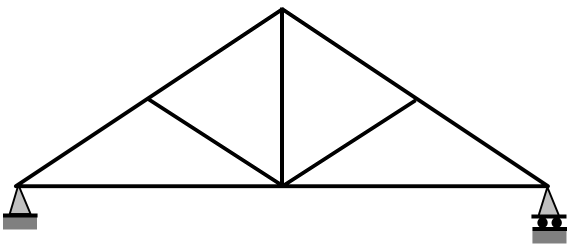
Figure 13 A statically determinate roof truss

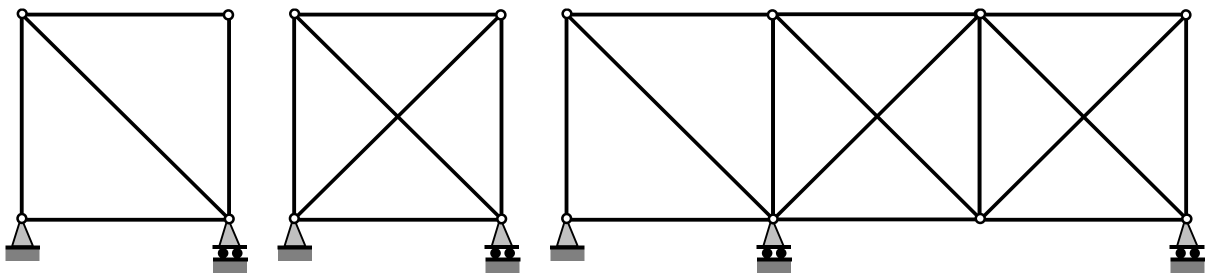

Figure 14 (a) Statically determinate truss, (b) Internally indeterminate, and (c) Both internally and externally indeterminate truss

A statically indeterminate truss can be analysed first by determining its DSI. Then the redundants (= DSI) are identified and they are removed to get the statically determinate *primary structure*. Then as in the case of beams and frames, trusses too can be analysed using:

$$\Delta_{i0} + \sum_{j=1}^{n} f_{ij} R_j = \Delta_i, \text{ for } i = 1 \text{ to } n$$

where Δ_{i0} is the displacement in the primary structure corresponding to the redundant R_i due to the applied loading, f_{ij} are the flexibility coefficients and Δ_i is the final displacement of the indeterminate structure at i along R_i. The following example demonstrates the method.

Exercise 5: Analyses and determine all the bar forces in the truss shown in Fig. 15 (a). The modulus of elasticity E is the same for all the members. The horizontal and vertical bars are of area A = 400 mm^2. The two diagonal braces have areas of 500 mm^2.

Here, the degree of static indeterminacy, DSI = 2. One internal (R_1) and one external redundancy (R_2). So we will consider the diagonal member AD and the horizontal reaction at B as the two redundants. Fig. 15 (b) shows the primary structure (the statically determinate one) with the two redundants. Fig. 15 (c) shows the primary structure subjected to the applied loading. All the bar forces are also shown in the figure. A unit tensile force applied along the bar AD (i.e. R_1 = 1) and the corresponding bar forces are shown in Fig. 15 (d). Fig. 15 (e) shows the case when H_B = 1 is applied.

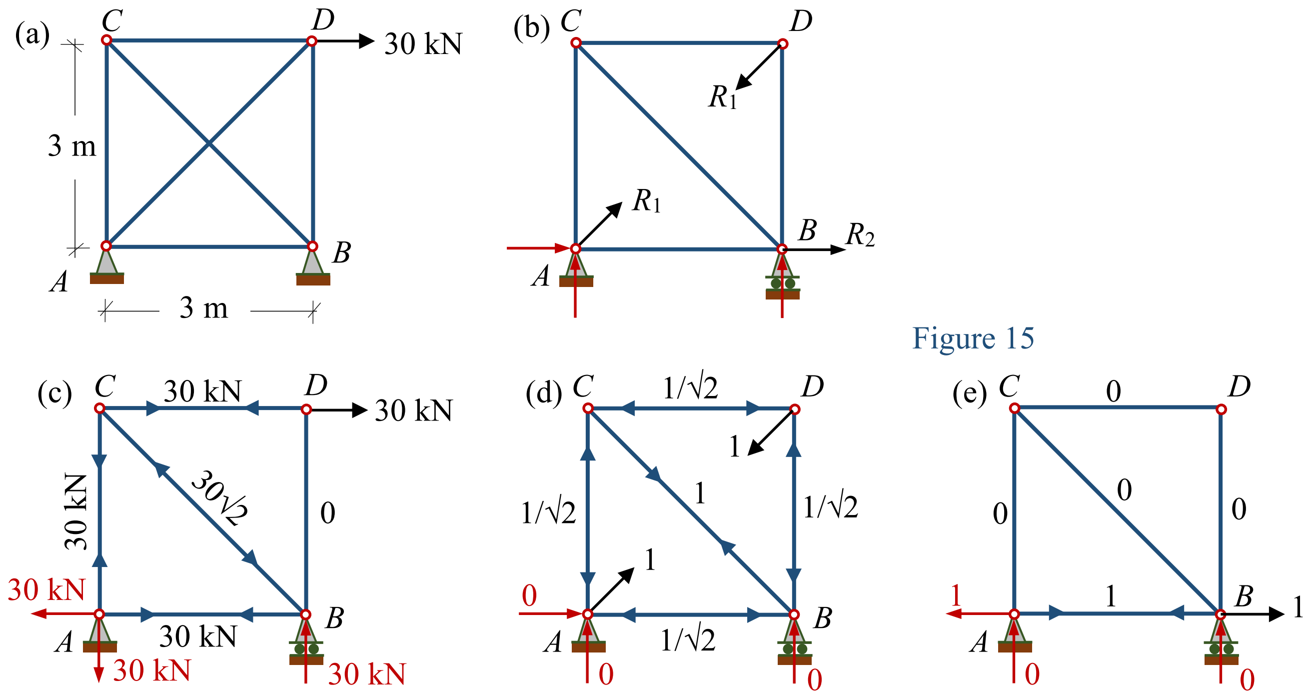

Figure 15

All the bar forces are shown in the table given below. Members are also indicated by numbers. Note that AE/L is a constant except for the two diagonals. For these two, area = 5/4A and length = √2 L.

Member		A	L	F_0	F_1	F_2	$F_0 F_1 L/AE$	$F_0 F_2 L/AE$	$F_1^2 L/AE$	$F_1 F_2 L/AE$	$F_2^2 L/AE$
1	*AC*	*A*	*L*	30	$-1/\sqrt{2}$	0	$-30/\sqrt{2}$	0	1/2	0	0
2	*CD*	*A*	*L*	30	$-1/\sqrt{2}$	0	$-30/\sqrt{2}$	0	1/2	0	0
3	*DB*	*A*	*L*	0	$-1/\sqrt{2}$	0	0	0	1/2	0	0
4	*AB*	*A*	*L*	30	$-1/\sqrt{2}$	1	$-30/\sqrt{2}$	30	1/2	$-1/\sqrt{2}$	1
5	*BC*	A^*	L^*	$-30\sqrt{2}$	1	0	–48	0	1.131	0	0
6	*AD*	A^*	L^*	0	1	0	0	0	1.131	0	0
						∑	–111.6396	30	4.26274	–0.7071	1
						AE/L ×	$\Delta_{10}=\uparrow$	$\Delta_{20}=\uparrow$	$f_{11}=\uparrow$	$f_{12}=\uparrow$	$f_{22}=\uparrow$

($A^* = 5A/4$; $L^* = \sqrt{2}L$; $L = 3$ m; $A = 400$ mm^2)

$$\begin{Bmatrix}\Delta_{10}\\ \Delta_{20}\end{Bmatrix} + \begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}\begin{Bmatrix}R_1\\ R_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix},$$

which leads to

$$\frac{L}{AE}\begin{bmatrix} 4.26274 & -0.7071\\ -0.7071 & 1\end{bmatrix}\begin{Bmatrix}R_1\\ R_2\end{Bmatrix} = \frac{L}{AE}\begin{Bmatrix}111.6396\\ -30\end{Bmatrix}$$

from which we get $R_1 = 24.032$ kN (tension), and $R_2 = -13.0068$ kN (from right to left).

Indicating each of the members by 1 to 6 as shown in the first column of the above table, the bar force in member i is given by: $F_i = F_{i0} + F_{i1} R_1 + F_{i2} R_2$. This follows from the principle of linear superposition.

Thus, $F_1 = F_{AC} = 30 - 1/\sqrt{2}\times 24.032 + 0 = 13.01$ kN $= F_2 = F_{CD}$ (why?)

$F_3 = -16.993$ kN; $F_4 = 0$; $F_5 = -18.394$ kN; $F_6 = 24.032$ kN. (verify all these)

It is a good idea to solve the same problem with two other redundants (say member *AB* and H_A or any other combination.