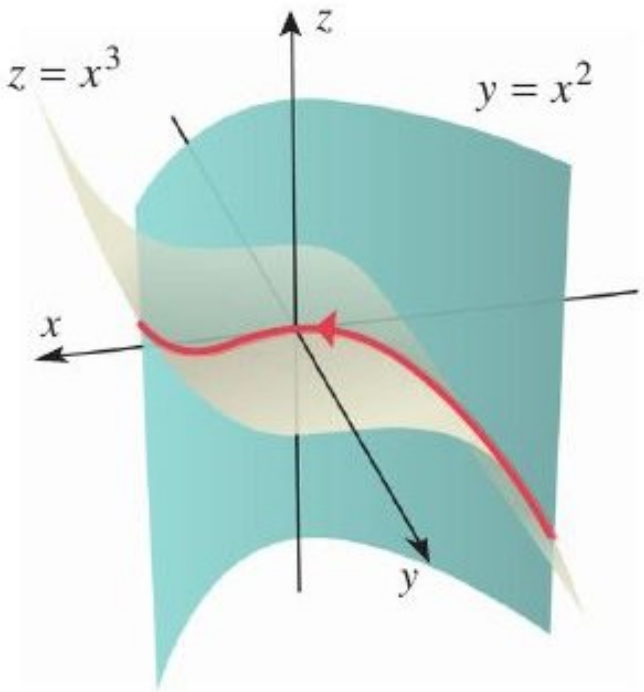
VECTOR CALCULUS - BASED ON ANTON, BEVINS

DEEPESH K P
(NIT Calicut)

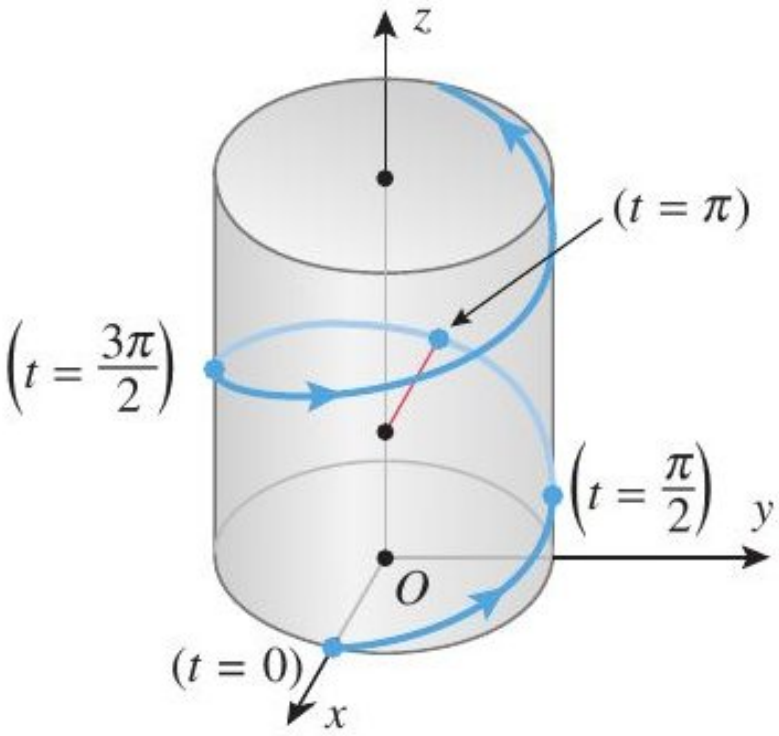
- $f(t) = t^2$ is a real valued function of one variable.
- $f : \mathbb{R} \to \mathbb{R}^2$ has codomain consisting of vectors.
- $f : \mathbb{R} \to \mathbb{R}^3$ also.
- Hence domain is 1-dimensional represented by a parameter.
- Codomain is a $2D/3D$ space, consisting of vectors.
- Eg: $f(t) = t^2\, i + sin\, t\, j + e^t\, k$ is a vector valued function (vvf)
- Image of $f(t) = x(t)\, i + y(t)\, j + z(t)\, k$ is a 3-D curve.
- $f(t) = (x(t), y(t), z(t))$ is also the same function.

Example $f : [-1, 1] \to \mathbb{R}^3$ by $f(t) = t\,i + t^2\,j + t^3\,k$



$$x = t,\ y = t^2,\ z = t^3$$

Example $f : [0, \infty) \to \mathbb{R}^3$ by $f(t) = a\cos t\, i + a\sin t\, j + ct\, k$



$$x = a\cos t,\quad y = a\sin t,\quad z = ct$$

Properties

- The 'graph' of a vvf is ploted with domain and codomain separately.
- If not specified, all possible values are taken for the parameter t (the natural domain).
- The component functions are represented as $x(t)$, $y(t)$ and $z(t)$.
- How do we sketch the graphical represenation of the image?

- The straight line segment joining (x_1, y_1, z_1) to (x_2, y_2, z_2).
- $\mathbf{r} : [0, 1] \to \mathbb{R}^3$ given by

$$\mathbf{r}(t) = \{x_1 + t(x_2 - x_1)\}i + \{y_1 + t(y_2 - y_1)\}j + \{z_1 + t(z_2 - z_1)\}k$$

- What happens if we take $t = 0$ or $t = 1$?
- What if t goes outside $[0, 1]$?
- What are $x(t)$, $y(t)$ and $z(t)$?
- It comes from the two point form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

- For $2D$ line, discard k component.

VECTOR FORM OF THE LINE

- Take $\mathbf{r}_1 = x_1\, i + y_1\, j + z_1\, k$ and $\mathbf{r}_2 = x_2\, i + y_2\, j + z_2\, k$
- $\mathbf{r} : [0, 1] \to \mathbb{R}^3$ given by

$$\mathbf{r}(t) = x_1\, i + y_1\, j + z_1\, k + t\{(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k\}$$

- This gives

$$\mathbf{r}(t) = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1),\ 0 \leq t \leq 1$$

- It is called vector form of the straightline.
  Example:
- Equation of line joining $(2, -3, 7)$ to $(1, 0, -15)$ is

$$\mathbf{r}(t) = 2i - 3j + 7k + t(-i + 3j - 22k),\ 0 \leq t \leq 1$$

- Parametric form $f(t) = (2 - t)i + (-3 + 3t)j + (7 - 22t)k$
- Component form $x = 2 - t,\ y = -3 + 3t,\ z = 7 - 22t$

- The circle parallel to the $XY-$plane (in 3D), radius a, center at $z_0 = (x_0, y_0, z_0)$ is given by $\mathbf{r} : [0, 2\pi) \to \mathbb{R}^3$

$$\mathbf{r}(t) = x_0 i + y_0 j + z_0 k + a(\cos t\, i + \sin t\, j) = z_0 + ae^{it}$$

- Here the parameter is taken as the angle.
- What if t goes outside $[0, 2\pi)$?
- Component form:
  $x(t) = x_0 + a\cos t,\ \ y(t) = y_0 + a\sin t,\ \ z(t) = z_0.$
- Note: $(x - x_0)^2 + (y - y_0)^2 = a^2$ is circle in XY-plane.

- Equation of a circle at height 4, radius 2 and center $(1, -3, 4)$ is ?

  (There are many circles in $3D$ with same center and radius. The most general equation is

  $$\mathbf{r}(t) = z_0 + a(\cos t\,\mathbf{u} + \sin t\,\mathbf{v}),$$

  u,v are orthogonal unit vectors which makes the circle slanted.)

Calculus of vvf: Limits, continuity, derivatives and integrals

Ordinary functions:

- $f(x) = \frac{x^2-4}{x-2}$ has limit 4 at $x = 2$.
- $\lim_{x \to a} f(x) = l$ means whenever $x \to a$, $f(x) \to l$ (calculate)
- Since domain is 1-dimensional we move in a line, has left/right directions only.
- Left limit: when $x \to a$ from left, $f(x) \to l_1$
- Right limit: when $x \to a$ from right, $f(x) \to l_2$
- If $l_1 = l_2$, we can say limit exists at a
- If limit exists and equals value, we say f is continuous at a.

$$\lim_{x \to a} f(x) = f(a).$$

- What about continuity of the example above? redefine it to make continuous!

Limits and continuity of Vector valued functions

- It is almost the same as ordinary functions.
- $f : \mathbb{R} \to \mathbb{R}^3$, domain is real number system.
- $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}_0$ means whenever $t \to a$ in the domain, the image values $\mathbf{r}(t) \to \mathbf{r}_0$.
- We can use the right limit - left limit idea and say that they are same/different.
- Suppose $\mathbf{f}(t) = x(t)i + y(t)j + z(t)k$, given in components.
- $\lim_{t \to a} \mathbf{r}(t)$ exists if and only if $\lim_{t \to a} x(t)$, $\lim_{t \to a} y(t)$, $\lim_{t \to a} z(t)$ each exists.
- $\lim_{t \to a} \mathbf{r}(t) = \lim_{t \to a} x(t)\mathbf{i} + \lim_{t \to a} y(t)\mathbf{j} + \lim_{t \to a} z(t)\mathbf{k}$.
- This is the easiest way to check.
- If limit exists and equals value, we say f is continuous at a point.

$$\lim_{t \to a} \mathbf{f}(t) = \mathbf{f}(a).$$

Definition of limit of VVF

12.2.1 DEFINITION Let $\mathbf{r}(t)$ be a vector-valued function that is defined for all t in some open interval containing the number a, except that $\mathbf{r}(t)$ need not be defined at a. We will write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

if and only if

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0$$

12.2.2 THEOREM

(*a*) *If* $\mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j}$, *then*

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t) \right\rangle = \lim_{t \to a} x(t)\mathbf{i} + \lim_{t \to a} y(t)\mathbf{j}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ *approaches a limiting vector as* t *approaches* a.

(*b*) *If* $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, *then*

$$\begin{aligned} \lim_{t \to a} \mathbf{r}(t) &= \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle \\ &= \lim_{t \to a} x(t)\mathbf{i} + \lim_{t \to a} y(t)\mathbf{j} + \lim_{t \to a} z(t)\mathbf{k} \end{aligned}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ *approaches a limiting vector as* t *approaches* a.

Examples

- $\lim_{t \to 1}(t^3\mathbf{i} + 2t^2\mathbf{j}) = \mathbf{i} + 2\mathbf{j}$
- $\lim_{t \to 0}(\frac{1}{t^2}\mathbf{i} + 2t^2\mathbf{j})$ does not exists as the first component does not have a limit at $t = 0$.
- $\lim_{t \to 2}(\frac{t^2 - 4}{t - 2}\mathbf{i} + 2t^2\mathbf{j}) = 4\mathbf{i} + 8\mathbf{j}$.
- Are these functions continuous?

- Derivative is a limit
- $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$, provided the limit exists!
- Rate of change of the function for small changes
- limit need not exist eg., $f(x) = |x|$ at 0
- Differentiable $\Rightarrow$ continuous $\Rightarrow$ Existance of limits
- Notations: $f'(a)$, $\frac{df}{dx}|_{x=a}$, $Df(a)$.
- Geometrically $f'(a)$ gives the slope of the tangent to the curve (graph) at the point $(a, f(a))$.

Derivative of VVF

- Similar to usual derivatives.
- If the limit of the rate of change of the function exists!
- $\lim_{t \to 0} \dfrac{\mathbf{f}(a+t) - \mathbf{f}(a)}{t} = \dfrac{d\mathbf{f}}{dt}(a)$
- It is just the component-wise derivative.
- We use $\mathbf{f}'(a)$.
- If $\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is differentiable,

$$\mathbf{f}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

Derivative of VVF

12.2.3 DEFINITION If $\mathbf{r}(t)$ is a vector-valued function, we define the ***derivative of*** $\mathbf{r}$ ***with respect to*** t to be the vector-valued function $\mathbf{r}'$ given by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{4}$$

The domain of $\mathbf{r}'$ consists of all values of t in the domain of $\mathbf{r}(t)$ for which the limit exists.

Properties of derivatives

12.2.6 THEOREM (*Rules of Differentiation*) *Let* $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, *and* $\mathbf{r}_2(t)$ *be differentiable vector-valued functions that are all in 2-space or all in 3-space, and let* $f(t)$ *be a differentiable real-valued function,* k *a scalar, and* $\mathbf{c}$ *a constant vector (that is, a vector whose value does not depend on* t*). Then the following rules of differentiation hold:*

(a) $\dfrac{d}{dt}[\mathbf{c}] = \mathbf{0}$

(b) $\dfrac{d}{dt}[k\mathbf{r}(t)] = k\dfrac{d}{dt}[\mathbf{r}(t)]$

(c) $\dfrac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] + \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(d) $\dfrac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] - \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(e) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\dfrac{d}{dt}[\mathbf{r}(t)] + \dfrac{d}{dt}[f(t)]\mathbf{r}(t)$

Here also

$$\int_a^b \bar{f}(t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} f_1(t_i)\Delta t_i \hat{i} + f_2(t_i)\Delta t_i \hat{j}$$

$$= \int_a^b f_1(t)dt\,\hat{i} + \int_a^b f_2(t)dt\,\hat{j}$$

It is the componentwise integral of the function! All properties are similar to the ordinary derivatives.

12.2.9 THEOREM (Rules of Integration) *Let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, and $\mathbf{r}_2(t)$ be vector-valued functions in 2-space or 3-space that are continuous on the interval $a \le t \le b$, and let k be a scalar. Then the following rules of integration hold:*

(a) $\displaystyle\int_a^b k\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt$

(b) $\displaystyle\int_a^b [\mathbf{r}_1(t) + \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt + \int_a^b \mathbf{r}_2(t)\,dt$

(c) $\displaystyle\int_a^b [\mathbf{r}_1(t) - \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt - \int_a^b \mathbf{r}_2(t)\,dt$

1–4 Find the limit.

1. $\lim_{t \to +\infty} \left\langle \frac{t^2+1}{3t^2+2}, \frac{1}{t} \right\rangle$

2. $\lim_{t \to 0^+} \left(\sqrt{t}\,\mathbf{i} + \frac{\sin t}{t}\,\mathbf{j} \right)$

3. $\lim_{t \to 2} (t\mathbf{i} - 3\mathbf{j} + t^2\mathbf{k})$

4. $\lim_{t \to 1} \left\langle \frac{3}{t^2}, \frac{\ln t}{t^2-1}, \sin 2t \right\rangle$

5–6 Determine whether $\mathbf{r}(t)$ is continuous at $t = 0$. Explain your reasoning.

5. (a) $\mathbf{r}(t) = 3 \sin t\,\mathbf{i} - 2t\mathbf{j}$ (b) $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{t}\mathbf{j} + t\mathbf{k}$

6. (a) $\mathbf{r}(t) = e^t\mathbf{i} + \mathbf{j} + \csc t\mathbf{k}$
   (b) $\mathbf{r}(t) = 5\mathbf{i} - \sqrt{3t+1}\,\mathbf{j} + e^{2t}\mathbf{k}$

7. Sketch the circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, and in each part draw the vector with its correct length.
   (a) $\mathbf{r}'(\pi/4)$ (b) $\mathbf{r}''(\pi)$ (c) $\mathbf{r}(2\pi) - \mathbf{r}(3\pi/2)$

8. Sketch the circle $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, and in each part draw the vector with its correct length.
   (a) $\mathbf{r}'(\pi/4)$ (b) $\mathbf{r}''(\pi)$ (c) $\mathbf{r}(2\pi) - \mathbf{r}(3\pi/2)$

9–10 Find $\mathbf{r}'(t)$.

9. $\mathbf{r}(t) = 4\mathbf{i} - \cos t\,\mathbf{j}$

10. $\mathbf{r}(t) = (\tan^{-1} t)\mathbf{i} + t \cos t\,\mathbf{j} - \sqrt{t}\,\mathbf{k}$

Properties of product of Derivatives of VVF

- $\frac{d}{dt}(\mathbf{c} \cdot \mathbf{r}(t)) = \mathbf{c} \cdot \frac{d}{dt}(\mathbf{r}(t))$
- $\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1(t) \cdot \frac{d}{dt}(\mathbf{r}_2(t)) + \frac{d}{dt}(\mathbf{r}_1(t)) \cdot \mathbf{r}_2(t)$
  - commutative
- $\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{r}_1(t) \times \frac{d}{dt}(\mathbf{r}_2(t)) + \frac{d}{dt}(\mathbf{r}_1(t)) \times \mathbf{r}_2(t)$
  - not commutative
- Proofs can be seen by taking components.

Note: Chain rule for composition: If $\mathbf{r} = \mathbf{r}(s)$ with $s = s(t)$, then

$$\frac{d}{dt}(\mathbf{r}(s)) = \frac{d}{ds}(\mathbf{r}(s))\frac{ds}{dt}$$

Ex: $\mathbf{r}(s) = s^3\hat{i} + 3s^2\hat{j}$ where $s = (t+1)^2$, then

$$\frac{d\mathbf{r}}{dt} = (3s^2\hat{i} + 6s\hat{j})2(t+1)$$

We make some standard assumptions here onwards:

- We consider only smooth (or piecewise smooth) curves

SMOOTH CURVE

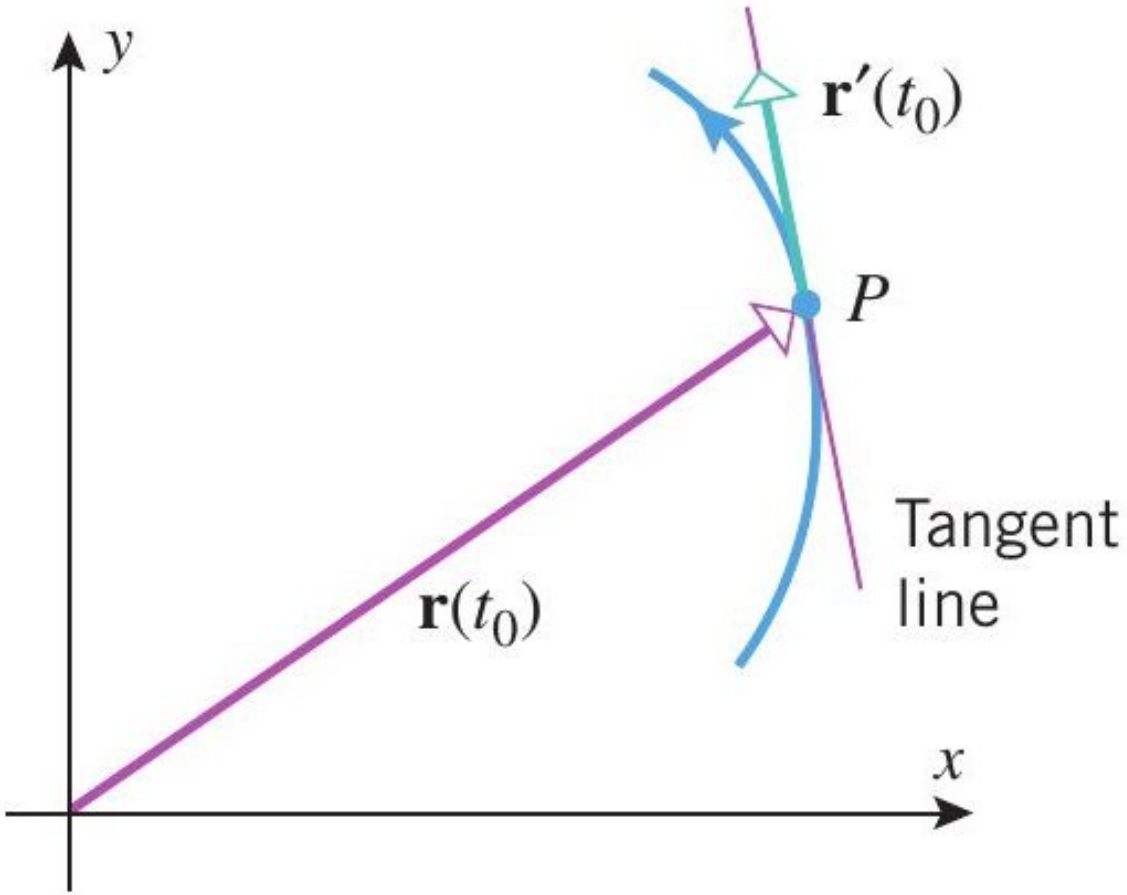
$\mathbf{r}(t)$ is smooth if differentiable, $\mathbf{r}'(t)$ continuous and $\mathbf{r}'(t) \neq 0$ at any t.

  - It ensures existance of tangents at each point
  - Tangent changes continuously.
  - particle always moves, velocity is never 0 till end

- The direction of $\mathbf{r}'(t)$ is always taken in the forward direction.

GEOMETRICAL IMPORTANCE OF DERIVATIVE OF VVF

- The derivative of $\mathbf{r}(t)$ at t_0 gives the slope of the tangent to the image curve at the position $\mathbf{r}(t_0)$.



GEOMETRICAL IMPORTANCE OF DERIVATIVE OF VVF

- From definition $\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)$ gives an secant.
- Multiplying by $\frac{1}{h}$ does not change the direction.
- Hence $\frac{\mathbf{r}(t_0+h)-\mathbf{r}(t_0)}{h}$ gives a secant.
- Taking limit as $h \to 0$, we get the tangent at $\mathbf{r}(t_0)$.

> **12.2.7 DEFINITION** Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to P (Figure 12.2.4). If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then we call $\mathbf{r}'(t_0)$ a ***tangent vector*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$, and we call the line through P that is parallel to the tangent vector the ***tangent line*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$.

- Equation of the tangent line to $\mathbf{r}(t)$ at t_0 is $\mathbf{r}(t_0) + t\,\mathbf{r}'(t_0)$

Unit tangent vector

- For $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, length $\|v\| = \sqrt{a^2 + b^2 + c^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- For any $\mathbf{v} \neq 0$, we know $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector in $\mathbf{v}$-direction.
- Hence the unit tangent vector function is $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$
- The angle between two vectors $\mathbf{u}$,$\mathbf{v}$ is given by $cos\,\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$
- Angle $\theta = cos^{-1}(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|})$
- Angle between two intersecting curves of vvf $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$?

Find the tangent vector, unit tangent vector and equation of the tangent line at $(1, 1, 4)$ to the curve

$$\mathbf{r}(t) = t^2\mathbf{i} + e^{t-1}\mathbf{j} + 4\mathbf{k}$$

- Here $t_0 = 1$.
- Tangent vector function is $\mathbf{r}'(t) = 2t\mathbf{i} + e^{t-1}\mathbf{j}$
- Tangent vector at $t = 1$ is $\mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}$
- The unit tangent vector at $t = 0$ is $\frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$
- Equation of tangent line is

$$\mathbf{r}(t) = (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) + t(2\mathbf{i} + \mathbf{j}), \qquad -\infty < t < \infty$$

Normal vector and Unit normal vector

- Two vectors are perpendicular if $\mathbf{u} \cdot \mathbf{v} = 0$
- For $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, length $\|v\|^2 = \mathbf{v} \cdot \mathbf{v}$
- For unit tangent vector function, $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$.
- Taking derivatives, we get $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$
- Normal vector is perpendicular to the tangent vector in the plane.
- $\mathbf{T}'(t)$ gives the normal vectors to the curve $\mathbf{r}(t)$.
- The unit normal vector is given by $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$

Find the unit tangent and unit normal to the curve, at $t = 0$, given by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$$

- Tangent vector function is $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$
- Tangent vector at $t = 0$ is $\mathbf{r}'(0) = 0\mathbf{i} + 1\mathbf{j}$
- The unit tangent vector is $\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t\mathbf{i} + \cos t\mathbf{j}}{1}$
- The unit tangent vector at $t = 0$ is $\frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \mathbf{j}$
- Normal vector is $\mathbf{N}(t) = \frac{-\cos t\mathbf{i} - \sin t\mathbf{j}}{1}$
- Unit normal vector is the same.
- At $t = 0$, the unit normal vector is $-\mathbf{i}$

Integrals using antiderivatives

- Suppose we know that $\frac{d}{dt}(\mathbf{F}(t) = \mathbf{f}(t)$
- Then $\int f(t)\,dt = \mathbf{F}(t) + \mathbf{C}$
- $\mathbf{C}$ is a constant vector!
- $\int_a^b f(t)\,dt = \mathbf{F}(b) - \mathbf{F}(a)$

▶ **Example 9** Find $\mathbf{r}(t)$ given that $\mathbf{r}'(t) = \langle 3, 2t \rangle$ and $\mathbf{r}(1) = \langle 2, 5 \rangle$.

Solution. Integrating $\mathbf{r}'(t)$ to obtain $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\, dt = \int \langle 3, 2t \rangle\, dt = \langle 3t, t^2 \rangle + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration. To find the value of $\mathbf{C}$ we substitute $t = 1$ and use the given value of $\mathbf{r}(1)$ to obtain

$$\mathbf{r}(1) = \langle 3, 1 \rangle + \mathbf{C} = \langle 2, 5 \rangle$$

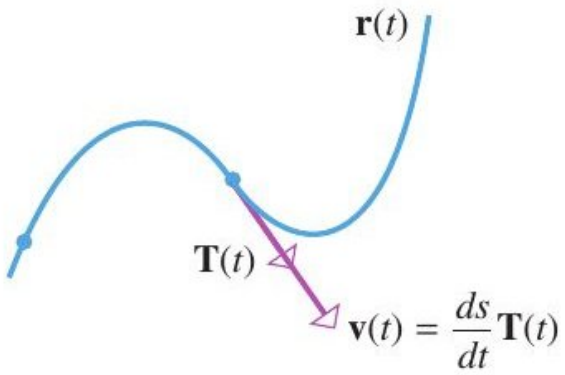
so that $\mathbf{C} = \langle -1, 4 \rangle$. Thus,

$$\mathbf{r}(t) = \langle 3t, t^2 \rangle + \langle -1, 4 \rangle = \langle 3t - 1, t^2 + 4 \rangle \quad \blacktriangleleft$$

Velocity Vector function

We can consider a curve as a function of time ($\mathbf{r} = \mathbf{r}(t)$). Also it can be treated as a function of distance ($\mathbf{r} = \mathbf{r}(s)$).

- Speed is defined as the rate of change of distance w.r.t. time $\frac{ds}{dt}$
- The direction of motion is the tangential direction (i.e. of $\mathbf{T}(t)$)
- Therefore velocity vector at time t is $\mathbf{v}(t) = \frac{ds}{dt}\mathbf{T}(t)$
- So magnitude of velocity vector is the speed (i.e., $\|v(t)\| = \frac{ds}{dt}$)



The length of the velocity vector is the speed of the particle, and the direction of the velocity vector is the direction of motion.

Velocity and acceleration

- Using chain rule, treating $\mathbf{r} = \mathbf{r}(s)$ and $s = s(t)$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\mathbf{T}(t)$$

- Hence $\mathbf{T} = \frac{d\mathbf{r}}{ds}$ also.
- Acceleration is the derivative of velocity $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$

12.6.1 DEFINITION If $\mathbf{r}(t)$ is the position function of a particle moving along a curve in 2-space or 3-space, then the ***instantaneous velocity***, ***instantaneous acceleration***, and ***instantaneous speed*** of the particle at time t are defined by

$$\text{velocity} = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \tag{2}$$

$$\text{acceleration} = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{3}$$

$$\text{speed} = \|\mathbf{v}(t)\| = \frac{ds}{dt} \tag{4}$$

Velocity and acceleration

FORMULAS FOR POSITION, VELOCITY, ACCELERATION, AND SPEED

	2-SPACE	3-SPACE
POSITION	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
VELOCITY	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$
ACCELERATION	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$
SPEED	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Displacement and distance travelled

Suppose a particle is travelling from P to Q as t moves from t_0 to t_1.

- The displacement is the direct measure from P to Q.

$$\Delta \mathbf{r} = \mathbf{r}(t_1) - \mathbf{r}(t_0).$$

- It is also the integral of velocity from t_0 to t_1.

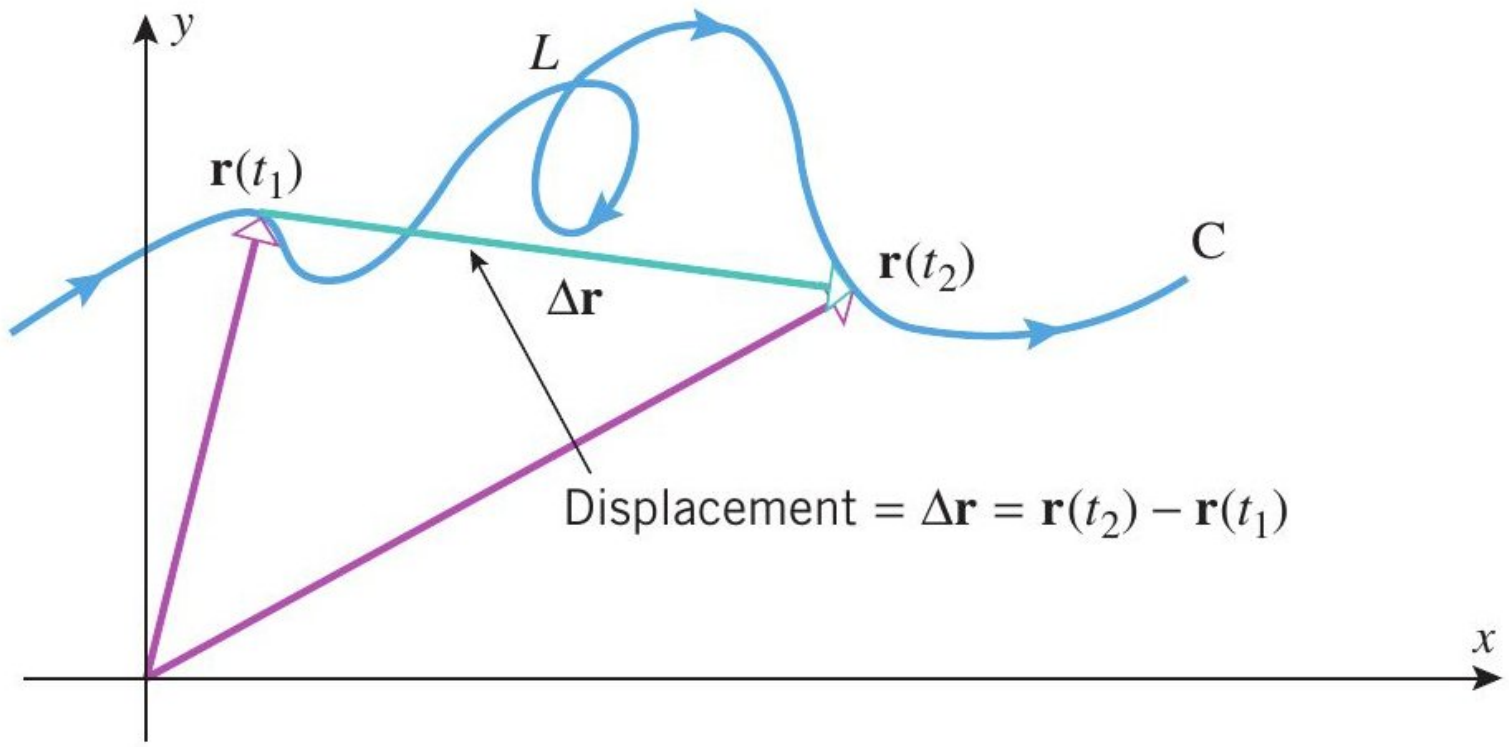
$$\Delta \mathbf{r} = \int_{t_0}^{t_1} \mathbf{r}'(t)\, dt = \mathbf{r}(t_1) - \mathbf{r}(t_0) = \int_{t_0}^{t_1} \mathbf{v}(t)\, dt$$

Distance travelled:

- Speed is the derivative of distance w.r.to t.
- Length of a curve $\mathbf{r}(t),\ t_0 \leq t \leq t_1$ is given by $\int_{t_0}^{t_1} \|\mathbf{r}'(t)\|\, dt$.
- This is the distance travelled by the particle:

$$s = \int_{t_0}^{t_1} \|\mathbf{r}'(t)\|\, dt = \int_{t_0}^{t_1} \|\mathbf{v}(t)\|\, dt$$

Distance and displacement



Velocity and acceleration

So, are the veclocity vector and tangent vector same?

- For velocity, we need the parameter to be time t!
- When time is the parameter, velocity is same as tangent vector.
- For any parameter, tangent vector is defined. But not velocity.

Is there any connection between normal vector and acceleration?

- Even if t is time, $\overline{a}$ need not be in the normal direction!
- Normal is the derivative of $\overline{T}$; whereas acceleration is the derivative of $\overline{r}'(t)$.
- When the tangent vector is a unit vector and the parameter is time, then only $\overline{a}$ coincides with the normal vector.
- We will see that acceleration has got both tangential and normal components, in general.

Arc length parametrization

A curve may be given as $\overline{r}(t)$, where t could be some parameter like time or twice of time or angle. The parameter could be time or any other parameter using which we express the function.

- Any curve will have a parametrization in terms of the arc length parameter.
- It is because, for each point we can associate a distance from the starting point, which could be taken as parameter!
- Since $\frac{ds}{dt} = \|\overline{v}(t)\|$, the relation between the given parameter t and the arc length parameter is given by

$$s = \int_{t_0}^{t} \|\overline{r}'(\tau)\| d\tau$$

- So $s = f(t)$ expresses the arc length in terms of t.
- If f invertible, $t = f^{-1}(s)$ expresses t in terms of s

A curve is given as $\bar{r}(t) = 3t^2\,\hat{i} + 4\,\hat{j}$, $0 \le t \le 1$, where t is some parameter. Find the arc length parametrization of this curve.

- $\|\bar{r}'(\tau)\| = \sqrt{36t^2} = 6t$. So

$$s = \int_{t_0}^{t} \|\bar{r}'(\tau)\| d\tau = \int_{t_0}^{t} 6\tau\, d\tau = 3t^2$$

- So $s = 3\,t^2 \Rightarrow t = \sqrt{\frac{s}{3}}$, $0 \le s \le 3$.
- The arc length parametrization is

$$\bar{f}(s) = 3\frac{s}{3}\,\hat{i} + 4\,\hat{j}.$$

Importance of time and arclength parametrisations

- Time t the the best parameter to study motion, velocity, acc.
- Arclength s is the best to study shape/geometry of the path.
- We could choose any point $\mathbf{r}(t_0)$ on the curve as base point for s
- When $t < t_0$, we get negative of arclength.
- Since smooth curve, $\frac{ds}{dt} > 0$. Hence s is invertible function of t.
- $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|v|}$.
- $\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{v}{|v|} = \mathbf{T}$

Curvature measures how fast the unit tangent vector **T** changes in direction (magnitude is contant) w.r.to s.

$$\text{curvature } \kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

- If κ large, curvature is more
- If $\kappa = 0$, it does not curve at all at the point.
- Using chain rule,
$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\right| \frac{1}{|v|}$$
- For a line, $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}_0$ or $\mathbf{r}(s) = \mathbf{r}_0 + \frac{s}{|v|}\mathbf{v}_0 \Rightarrow \mathbf{T} = 1,\ \kappa = 0.$
- For a circle $\mathbf{r}(t) = a\cos t\hat{i} + a\sin t\hat{j}$, $|v| = a$, $\left|\frac{d\mathbf{T}}{dt}\right| = 1 \Rightarrow \kappa = \frac{1}{a}$

Since $\mathbf{T}$ has unit length, $\frac{d\mathbf{T}}{ds} \perp \mathbf{T}$ and so

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \frac{\frac{d\mathbf{T}}{ds}}{\kappa}$$

This means

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}.$$

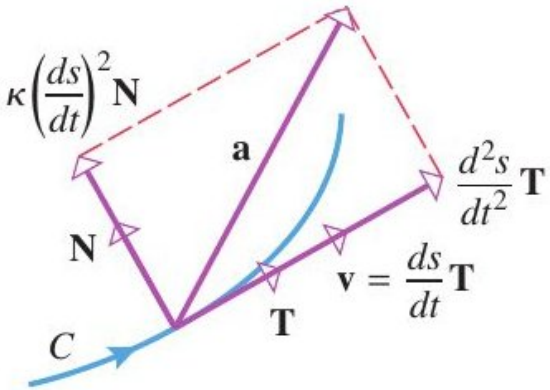
- Circle of curvature is the circle that has same tangent as that of the curve at the point.
- Also called osculating circle
- Since for a circle, curvature is $\kappa = \frac{1}{a}$, if the curve has curvature κ, radius of the circle is $\frac{1}{\kappa}$

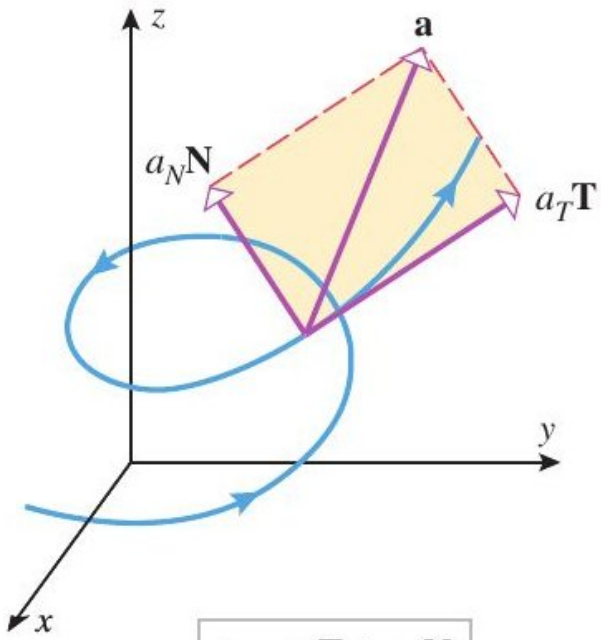
Acceleration components

12.6.2 THEOREM *If a particle moves along a smooth curve C in 2-space or 3-space, then at each point on the curve velocity and acceleration vectors can be written as*

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T} \qquad \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \tag{10–11}$$

where s is an arc length parameter for the curve, and $\mathbf{T}$, $\mathbf{N}$, and κ denote the unit tangent vector, unit normal vector, and curvature at the point (Figure 12.6.4).





$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

Resolving Acceleration

$$\begin{aligned}
\mathbf{a} = \frac{d}{dt}\left(\frac{ds}{dt}\mathbf{T}\right) &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\frac{d\mathbf{T}}{ds} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\kappa\mathbf{N}
\end{aligned}$$

Resolution of Acceleration

We have resolved acceleration to components: $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$

- Taking dot product with $\mathbf{T}$,

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$

- Taking dot product with $\mathbf{N}$,

$$a_N = \mathbf{a} \cdot \mathbf{N}$$

.

- Also, $|\mathbf{a}|^2 = {a_T}^2 + {a_N}^2$
- Taking modulus of cross product with $\mathbf{T}$,

$$a_N = \|\mathbf{a} \times \mathbf{T}\| = \frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|}$$

- Curvature is obtained from $a_N = \kappa(\frac{ds}{dt})^2$. So

$$\kappa = \frac{a_N}{(\frac{ds}{dt})^2} = \frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|^3}$$

12.6.3 THEOREM *If a particle moves along a smooth curve C in 2-space or 3-space, then at each point on the curve the velocity* $\mathbf{v}$ *and the acceleration* $\mathbf{a}$ *are related to* a_T, a_N, *and* κ *by the formulas*

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} \qquad a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} \qquad \kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} \tag{15–17}$$

$$a_T = \|\mathbf{a}\| \cos\theta \quad \text{and} \quad a_N = \|\mathbf{a}\| \sin\theta$$

from which we obtain

$$a_T = \|\mathbf{a}\| \cos\theta = \frac{\|\mathbf{v}\| \|\mathbf{a}\| \cos\theta}{\|\mathbf{v}\|} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

$$a_N = \|\mathbf{a}\| \sin\theta = \frac{\|\mathbf{v}\| \|\mathbf{a}\| \sin\theta}{\|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

$$\kappa = \frac{a_N}{(ds/dt)^2} = \frac{a_N}{\|\mathbf{v}\|^2} = \frac{1}{\|\mathbf{v}\|^2} \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

▶ **Example 1** A particle moves along a circular path in such a way that its x- and y-coordinates at time t are

$$x = 2\cos t, \quad y = 2\sin t$$

(a) Find the instantaneous velocity and speed of the particle at time t.

(b) Sketch the path of the particle, and show the position and velocity vectors at time $t = \pi/4$ with the velocity vector drawn so that its initial point is at the tip of the position vector.

(c) Show that at each instant the acceleration vector is perpendicular to the velocity vector.

▶ **Example 2** A particle moves through 3-space in such a way that its velocity is

$$\mathbf{v}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$$

Find the coordinates of the particle at time $t = 1$ given that the particle is at the point $(-1, 2, 4)$ at time $t = 0$.

▶ **Example 3** Suppose that a particle moves along a circular helix in 3-space so that its position vector at time t is

$$\mathbf{r}(t) = (4\cos \pi t)\mathbf{i} + (4\sin \pi t)\mathbf{j} + t\mathbf{k}$$

Find the distance traveled and the displacement of the particle during the time interval $1 \leq t \leq 5$.

▶ **Example 4** Suppose that a particle moves through 3-space so that its position vector at time t is

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

(The path is the twisted cubic shown in Figure 12.1.5.)

(a) Find the scalar tangential and normal components of acceleration at time t.

(b) Find the scalar tangential and normal components of acceleration at time $t = 1$.

(c) Find the vector tangential and normal components of acceleration at time $t = 1$.

(d) Find the curvature of the path at the point where the particle is located at time $t = 1$.

Binormal vector

Consider a curve in $3D$ space given by $\mathbf{r}(t)$.

- Apart from the tangent and normal vector, there is a vector perpendicular to both **T** and **N**.
- This vector is called the binormal vector.

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

so that (T, N, B) form a right handed system.

- 3 directions: Forward (tangential), twinning (normal) and twisting (binormal) directions
- It is called a Frenet frame or TNB frame.
- The base plane or **TN**-plane is called osculating plane
- The **NB**-plane is called normal plane (given by normal and binormal)
- The **BT**-plane is called rectifying plane.

Torsion τ

- It is similar to curvature κ (how fast **N** changes)
- How fast **B** changes w.r.to s?
- $\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$, since $\frac{d\mathbf{T}}{ds} \times \mathbf{N} = 0$, being parallel.
- $\Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{T}$, and $\frac{d\mathbf{B}}{ds} \perp \mathbf{B}$, **B** being of constant length.
- $\Rightarrow \frac{d\mathbf{B}}{ds}$ is parallel to **N**, giving $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$
- So $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$.
- So torsion is the negative of how fast **B** changes w.r.to s.
- The change in the normal vector reflects the change in the plane to which it is normal
- τ tells how fast the osculating plane changes, and κ tells how fast the normal plane changes

An application - path of a projectile

A projectile is fired from ground at time $t = 0$, with initial velocity $\mathbf{v}_0$. Assume only gravity acts on it which reduses the velocity.

- What is the position vector?
- What is the maximum height it will travel?
- What time it will hit the ground?
- How much distance it will travel?
  - Newton's law says mass·acceleration=effective force
  - $m\frac{d^2\mathbf{r}}{dt} = -mg\hat{j} \Rightarrow \frac{d^2\mathbf{r}}{dt} = -g\hat{j}$
  - integrate and use initial conditions $\mathbf{r}(0) = 0$, $\mathbf{r}'(0) = \mathbf{v}_0$
  - position vector $\mathbf{r}(t) = -g\frac{t^2}{2}\hat{j} + \mathbf{v}_0 t$
  - If fired with angle α, $\mathbf{v}_0 = |\mathbf{v}_0|\cos\alpha\hat{i} + |\mathbf{v}_0|\sin\alpha\hat{j}$

$$\text{Position vector is } \mathbf{r}(t) = v\cos\alpha t\,\hat{i} + (v\sin\alpha t - g\frac{t^2}{2})\hat{j}$$

*$|\mathbf{v}_0| = v$

An application - path of a projectile

The maximum height it goes y_{max} happens when $\mathbf{v}_{vertical} = 0$.

- $\mathbf{r}'(t) \cdot \hat{j} = v \sin\alpha - gt = 0 \Rightarrow t = \frac{v \sin\alpha}{g}$
- $y_{max} = \frac{1}{2}\frac{v^2 \sin^2\alpha}{g}$

Time it hits ground will have veritical part of $\mathbf{r}(t) = 0$

- $v \sin\alpha t = \frac{1}{2} g t^2 \Rightarrow t = 0,\ t = \frac{2v \sin\alpha}{g}$
- so it hits ground when $t = \frac{2v \sin\alpha}{g}$.

The range it travels is given by horizontal part of **r(t)** at $t = \frac{2v \sin\alpha}{g}$

- Horizonatal Range is $v \cos\alpha \frac{2v \sin\alpha}{g} = \frac{v^2}{g} \sin 2\alpha$.

Curvature for a curve $y = f(x)$

A curve $y = f(x)$ can be viewed as $(x, f(x))$.

- Think of the curve as $\mathbf{r}(t) = t\,\hat{i} + f(t)\,\hat{j}$.
- $\mathbf{v} = \hat{i} + f'(t)\hat{j}$, $\mathbf{a} = f''(t)\hat{j}$

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(t)|}{(1 + f'(t)^2)^{\frac{3}{2}}}$$

Hence the curvature is

$$\kappa = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}}$$

Parameter vs Variable

It depends on the situation. A variable can change within the context of the given situation (problem). But a parameter is fixed during a problem. By changing values of parameters, we get different problems.

- In $y = mx + c$, m, c are parameters.
- By changing m, c we get different lines.
- For each fixed m, c, the variable x can change in the domain.

Also in vector calculus, the parameters are coming from a different set (not from the given domain). The variables in a vector valued function changes in the domain.

THANK YOU