Isoparametric Triangles and Tetrahedra

Introduction:

- Triangles with curved edges and tetrahedra with curved faces can be developed
- Special numerical integrations are necessary
- Elements with straight edges and uniform spacing of edge nodes is a special case, and can be integrated exactly using special formulae

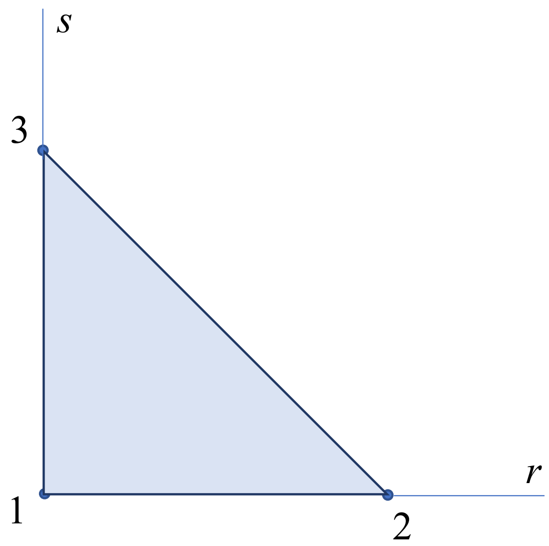
Linear Triangle:

- Natural coordinates r and s are used
- Interpolation polynomials are given by

$$N_1 = 1 - r - s$$

$$N_2 = r; \quad N_3 = s$$

Node	(r, s)
1	0,0
2	1,0
3	0,1



Isoparametric formulation

$$x = \sum N_i x_i \, ; \quad y = \sum N_i y_i$$

$$u = \sum N_i u_i \, ; \quad v = \sum N_i v_i$$

$$\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix}$$

[J]

$$[J] = \begin{bmatrix} \Sigma N_{i,r} x_i & \Sigma N_{i,r} y_i \\ \Sigma N_{i,s} x_i & \Sigma N_{i,s} y \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} x_{21} & y_{21} \\ x_{31} & y_{31} \end{bmatrix}; \quad \begin{matrix} x_{ij} = x_i - x_j \\ y_{ij} = y_i - y_j \end{matrix}$$

| J | = 2 A, A is the area of the triangle

A is positive, if nodes are numbered counter-clockwise around the triangle. Then [B] can be calculated much the same way as was done for isoparametric quadrilateral elements

$$[\,B] = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}; \qquad [k^e] = \int B^T D B \; t \; dA = B^T D B \; t \; A$$

Quadratic Triangle:

- Natural coordinates r and s are used
- Interpolation polynomials are given by

$$N_1 = (1 - r - s)\,(1 - 2r - 2s)$$

$$N_2 = r\,(2r - 1)$$

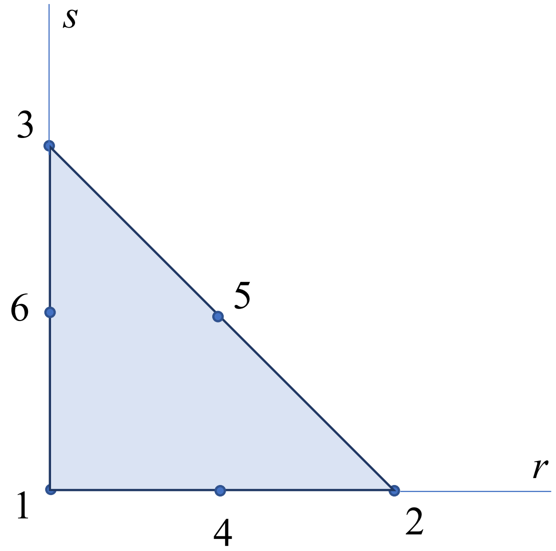
$$N_3 = s\,(2s - 1)$$

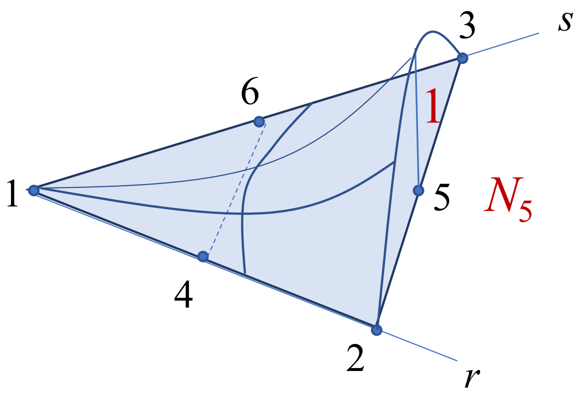
$$N_4 = 4r\,(1 - r - s)$$

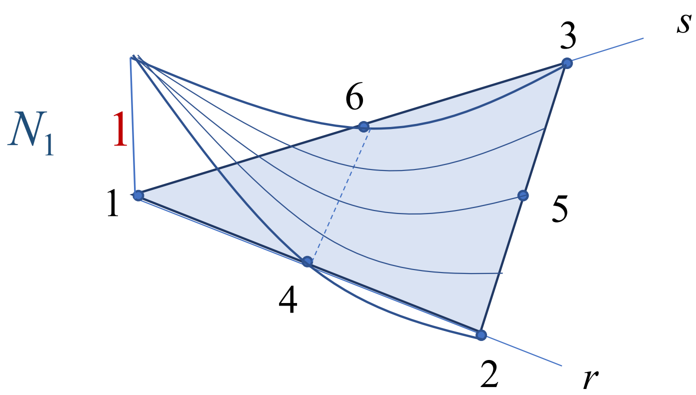
$$N_5 = 4rs$$

$$N_6 = 4s\,(1 - r - s)$$

Node	(r, s)
1	0,0
2	1,0
3	0,1
4	½, 0
5	½, ½
6	0, ½







- Here, for example, J_{11} is given by

$$J_{11} = \frac{\partial x}{\partial r} = (4r + 4s - 3)\, x_1 + (4r - 1)\, x_2 - 4(2r + s - 1)\, x_4 + 4s\, x_5 - 4s\, x_6$$

The multiplier of x_3 is zero. The other terms of J_{ij} are obtained along similar lines.

Numerical Integration

- In a distorted triangle with curved edges and nonuniformly spaced side nodes, J is not a constant
- The terms in integrands are ratios of polynomials
- Numerical integration is the only viable alternative
- A variety of numerical quadrature schemes are available

$$\int_A \phi \, dA = \sum_{i=1}^{n} \phi_i \, J_i \, W_i, \text{ where } J_i = \frac{1}{2} |\, \mathbf{J} \,|_i$$

The factor ½ appears because the area of the mapped triangle in *rs*-plane is ½

No. of Points	Figure	Degree of Precision*	Coordinates (r_i, s_i)	Weights W_i
1	a	1	$(\frac{1}{3}, \frac{1}{3})$	1.0
3	b	2	$(\frac{2}{3}, \frac{1}{6})$, $(\frac{1}{6}, \frac{1}{6})$, $(\frac{1}{6}, \frac{2}{3})$	$\frac{1}{3}$
3	c	2	$(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$	$\frac{1}{3}$
4	d	3	$(\frac{1}{3}, \frac{1}{3})$ $(\frac{3}{5}, \frac{1}{5})$, $(\frac{1}{5}, \frac{1}{5})$, $(\frac{1}{5}, \frac{3}{5})$	27/48 25/48

* Degree of highest polynomial terms in r and s that are integrated exactly by this formula. Specifically, a rule with degree of precision k exactly integrates all terms of the form $r^l\, s^m$ when $l + m \leq k$, and is inexact for at least one term $r^l\, s^m$ when $l + m > k$

- By retaining only vertex nodes, we get a four-noded linear tetrahedral element with shape functions:

$$N_1 = 1 - r - s - t$$

$$N_2 = r\,; \quad N_3 = s\,; N_4 = t$$

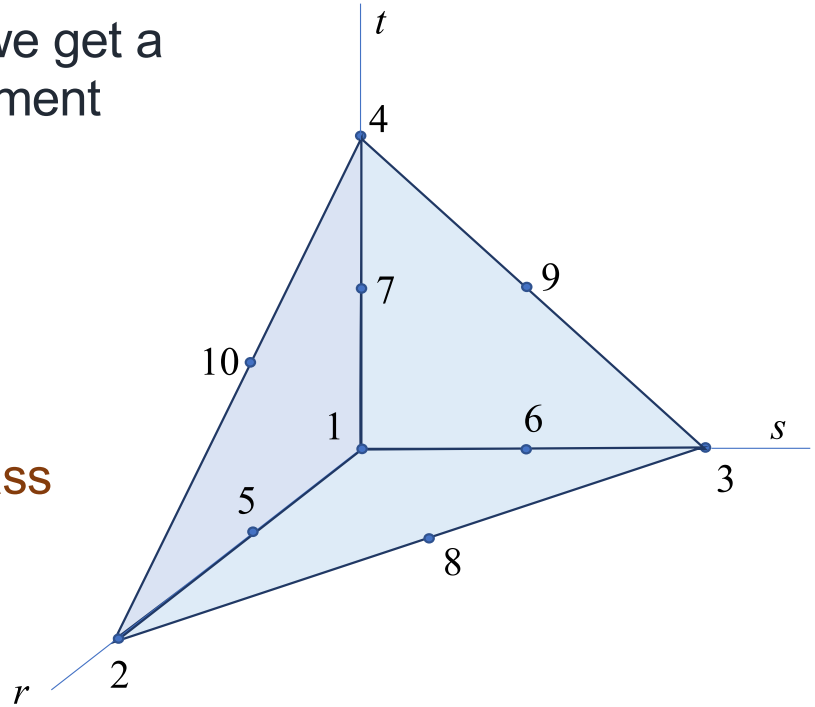
- $N_i = 0$ is the equation of the tetrahedron face that doesn't pass through node i
- For the 10-noded quadratic tetrahedron, each N_i is the product of two face functions:

$$N_1 = (1 - r - s - t)\,(1 - 2r - 2s - 2t)$$

$$N_2 = r\,(2r - 1); \; N_3 = s\,(2s - 1); \; N_4 = t\,(2t - 1)$$

$$N_5 = 4r\,(1 - r - s - t); \; N_6 = 4s\,(1 - r - s - t); \; N_7 = 4t\,(1 - r - s - t)$$

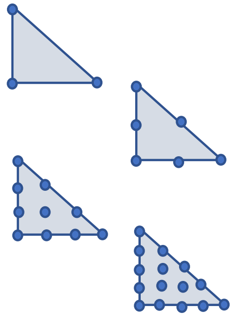
$$N_8 = 4rs; \; N_9 = 4st; \; N_{10} = 4tr$$



Node	(r, s, t)
1	0, 0, 0
2	1, 0, 0
3	0, 1, 0
4	0, 0, 1
5	½, 0, 0
6	0, ½, 0
7	0, 0, ½
8	½, ½, 0
9	0, ½, ½
10	½, 0, ½

- Higher order tetrahedra have internal nodes like higher order triangles

Pascal Triangle	p	N
1	0	
$x \quad y$	1	3
$x^2 \quad xy \quad y^2$	2	6
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3	10
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	4	15



$N = (p + 1)\,(p + 2)/2,$

N, no. of nodes = no. of terms

p, degree of polynomial

For tetrahedra,

$N = (p + 1)(p + 2)\,(p + 3)/6$

Isoparametric tetrahedra: This is similar to the case of triangular elements

$x = \sum N_i\, x_i\,; \quad y = \sum N_i\, y_i; \quad z = \sum N_i\, z_i$

$u = \sum N_i\, u_i\,; \quad v = \sum N_i\, v_i; \quad w = \sum N_i\, w_i$

Numerical Integration: Let $\phi = \phi(r, s, t)$ represent a single coefficient of k^e. Then

$\int_V \phi\, dV = \sum \phi_i\, J_i\, W_i$, where $J_i = ⅙\, |[\,J\,]|$; (⅙ is the volume of the mapped tetrahedron

See Table 7.4-2 (p268) of RD Cook et al. Concepts and Applications of FEA (4th edition)

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