ZZU101 ENGINEERING MECHANICS II—DYNAMICS

Text Book: I.H. Shames, *"Engineering Mechanics—Statics and Dynamics"*, 4^{th} Edition, Prentice Hall Inc, New Jersey, 1996.

MODULE 1

Kinematics of a Particle—Simple Relative Motion

Dynamics has two branches—*kinematics* and *kinetics* (or *dynamics*)

- *Kinematics* deals with the study of motion of particles and rigid bodies without consideration of the forces causing the motion.
- It is the study of *geometry* of motion.
- *Kinematics* needs to be mastered before attempting to learn *Dynamics (or Kinetics).*

Dynamics
- of a single particle
- of a system of particles
- of rigid bodies
- of a system of rigid bodies
- of a deformable body

The *particle* idealisation is valid if we are interested in the motion of the centre of mass alone and not in the size or orientation of the body during its motion.

Part A: General Notions

Differentiation of a Vector with Respect to Time

For a scalar, we have

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0}\left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right].$$

In the above, df/dt is another function of time, and can be further differentiated any number of times for suitable functions $f(t)$ to get the higher derivatives.

In the case of a vector, the variation in time may be due to:

- a change in magnitude,
- a change in direction, or
- a change in both.

Thus, the time derivative of a vector function $\mathbf{F}(t)$ is, by definition,

$$\frac{d\mathbf{F}}{dt} = \lim_{\Delta t \to 0}\left[\frac{\mathbf{F}(t+\Delta t) - \mathbf{F}(t)}{\Delta t}\right]$$

- If $\mathbf{F}$ has no change in direction during the time interval, the above definition is similar to the case of a scalar function.
- When $\mathbf{F}$ changes in direction, $d\mathbf{F}/dt$ will be different in both magnitude and direction from $\mathbf{F}$.

Consider a particle moving along a curvilinear path as shown in Fig. 1. The rate of change of position vector $\mathbf{r}(t)$ with respect to time is defined as the velocity vector $\mathbf{V}$ relative to the same frame of reference xyz as that of $\mathbf{r}$. Thus

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0}\left[\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}\right].$$

Or

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta\mathbf{r}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(\frac{\Delta\mathbf{r}}{\Delta s}\frac{\Delta s}{\Delta t}\right).$$

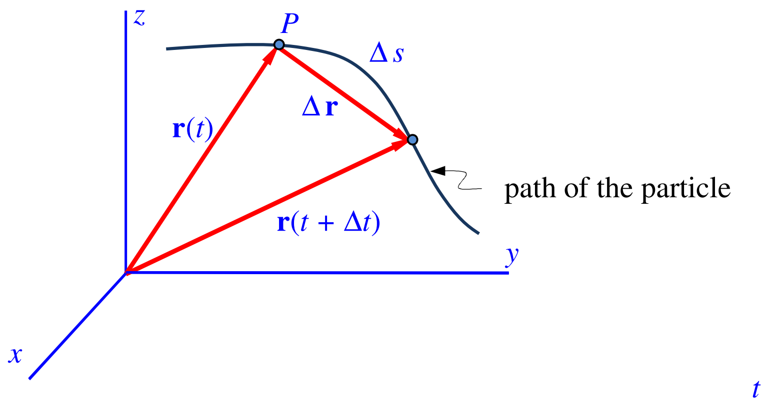

Figure 1

In the above, s is the arc length. As $\Delta t \to 0$, the direction of $\Delta\mathbf{r}$ approaches the tangent direction of the path at position $\mathbf{r}(t)$ and approaches Δs in magnitude.
Hence, as $\Delta t \to 0$

$$\frac{\Delta\mathbf{r}}{\Delta s} = \frac{\Delta s\, \boldsymbol{\varepsilon}_t}{\Delta s} = \boldsymbol{\varepsilon}_t\,,$$

where $\boldsymbol{\varepsilon}_t$ is the unit vector tangential to the trajectory. Thus

$$\frac{d\mathbf{r}}{ds} = \boldsymbol{\varepsilon}_t\,.$$

Hence

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\boldsymbol{\varepsilon}_t\,.$$

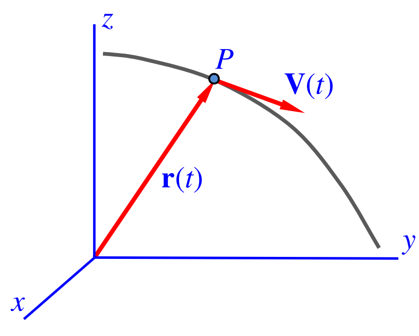

Figure 2

That is, $d\,\mathbf{r}/dt$ is a vector with magnitude equal to the *speed* of the particle and direction tangential to the path. The angle between $\mathbf{r}$ and $\mathbf{V}$ could be =90°, <90° or >90° as depicted in Fig. 2. The acceleration vector $\boldsymbol{a}$ is given by

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{r}}{dt^2}.$$

Part: B Velocity and Acceleration Calculation

Vectors can be expressed in many ways. We can use

- rectangular components,
- cylindrical polar components,
- spherical polar components, or
- path variables.

We shall consider rectangular, cylindrical polar, and path variables in what follows.

RECTANGULAR COMPONENTS

If **r** is given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Then the velocity vector, **V**(t) can be obtained as

$$\mathbf{V}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \text{ or } \mathbf{V}(t) = \dot{x}\dot{\mathbf{t}} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}.$$

The acceleration vector **a**(t) is given by

$$\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}.$$

Now, if **a**(t) is known in term of its components as given above, we can integrate its components to obtain the components of the velocity vector. Thus

$$V_x(t) = \int x(t)\, dt + C_1,$$

where C_1 is the constant of integration; C_1, can be determined if V_x is known at any particular time instance, e.g. at $t = t_0$. Integrating the above once again, we obtain

$$x(t) = \int \left\lfloor \int x \ dt \right\rfloor dt + C_1 + C_2.$$

Knowing x at $t = t_{0,}$ we can determine C.

Ex: 1 The pins A and B must always remain in the vertical slot of yoke C, which moves to the right with a constant speed of 2 m/s as shown in Fig. 3. The pins cannot leave the elliptic slot with the semi-major and minor axes of length 3 m and 2 m respectively. (a) What is the speed with which the pins approach each other when the yoke slot is at $x = 2$ m? (b) What is rate of change of speed toward each other when the yoke slot is at $x = 2$ m?

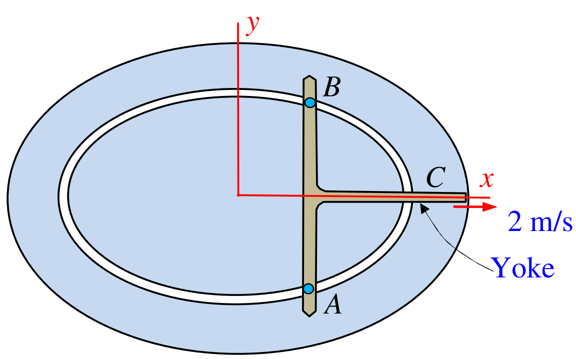

Figure 3

The equation of the ellipse is

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1 \qquad \textbf{[A]}$$

At any time t, x (t) and y (t) must satisfy [A]. Also, $\dot{x}(t)$ and $\dot{y}(t)$ must be such that the pin B moves in the elliptic slot at all time. Now, differentiating [A], we obtain

$$\frac{2x\dot{x}}{9}+\frac{2y\dot{y}}{4}=0 \qquad \textbf{[B]}$$

Thus, $x, \dot{x}, y$ and $\dot{y}$ must satisfy [B] for all time.

Now, $\dot{x} = 2$ m/s (as the yoke moves with this speed). When $x = 2$, from [A] we get: $y = 1.4907$ m. Hence, from [B], we obtain $\dot{y} = -1.1926$ m/s. This is the speed with which pin B moves. The pin A moves with the same speed in the opposite direction. Hence, the speed with which the pins approach each other is 2×1.1926 = <u>2.3851</u> m/s.

In order to get the acceleration of the pins, differentiate [B] once again to obtain

$$\frac{2}{9}\left(\dot{x}^2+x\ddot{x}\right)+\frac{1}{2}\left(\dot{y}^2+y\ddot{y}\right)=0 \qquad \text{[C]}$$

The accelerations $\ddot{x}$ and $\ddot{y}$ must satisfy [C]. Since the yoke moves with a constant speed, $\ddot{x} = 0$. Thus, we get $\ddot{y}_B$ from

$$\frac{2}{9}\left(2^2+0\right)+\frac{1}{2}\left((-1.1926)^2+1.4907\,\ddot{y}\right)=0.$$

Hence, $\ddot{y} = -0.2385$ m/s^2. Therefore, the rate of change of speed toward each other when the yoke slot is at x = 2m is = <u>0.477</u> m/s^2.

<u>Home work</u>: If at x = 2 m, the yoke has a velocity of 2 m/s and an acceleration of 1 m/s^2, find the above.

Motion of Projectiles

If we ignore air resistance, in the motion of a particle near the earth's surface, we have $\ddot{y}(t) = -g = -9.81$ m/s^2 (or 32.2 ft/s^2). Also $\ddot{x} = \ddot{z} = 0$. On integrating these accelerations, we can determine the position and velocity of these particles.

Ex: 2 A shell is fired at an elevation of 200 m above a plane as shown in Fig. 4. The angle of firing is 15^o with respect to the horizontal. If the initial velocity of the shell as it is being fired is 1000 m/s, determine the distance at which the shell hits the ground. Neglect the friction of the air. What is the maximum height to which the shell rises? What is the trajectory of the shell?

$$\ddot{y}(t)=\frac{dV_y}{dt}=-g=-9.81\,\text{m/s}^2, \qquad \text{[A]}$$

$$\ddot{x}(t)=\frac{dV_x}{dt}=0, \qquad \text{[B]}$$

and $z = \dot{z} = \ddot{z} = 0$ as the motion is coplanar. From [A], we obtain

$$V_y=(t)=-9.81t+A.$$

and from [B]

$$V_x(t)=B.$$

At t = 0, $V_x(0) = 1000\cos 15 = B$ i.e. B = <u>965.926</u> m/s. And V_y (0) = 1000 sin 15 = A. That is, A = <u>258.819</u> m/s.

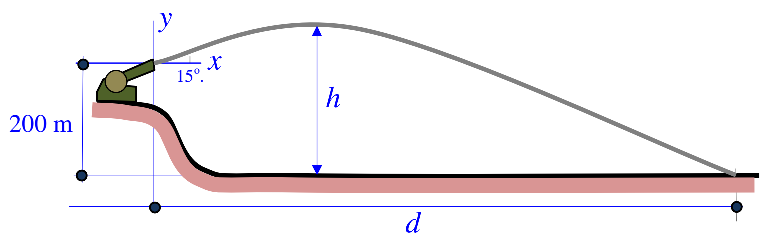

Figure 4

Thus, $V_x(t) = 965.926$ m/s. And $V_y(t) = 258.819 - 9.81$ t m/s. Hence, $x(t) = 965.926\,t + C$. And

$$y(t) = 258.819\,t - 9.81\,t^2/2 + D \qquad [C]$$

At $t = 0$, $x = 0$, $\Rightarrow C = 0$. At $t = 0$, $x = 0 \Rightarrow D = 0$.

To find "d"; first find "t" to reach $x = d$, $y = -200$. That is

$$-200 = 258.819t - \frac{9.81}{2}t^2,$$

from which we get

$$t = \frac{-258.819 \pm \sqrt{258.819^2 - 4\left(\frac{-9.81}{2}\right)200}}{-9.81} = \underline{53.528\text{ s}}.$$

Hence, $d = 965.926(53.528)/10^3 = \underline{51.704}$ km.

To find y_{max}, we first find "t" corresponding to $V_y = 0$.

$$0 = 258.819 - 9.81\,t. \quad \therefore t = \underline{26.38\text{ s}}.$$

Hence, $y_{max} = 258.819\,t - 9.81\,t^2/2 = \underline{3414.234}$ m $= \underline{3.414}$ km.

From [C], eliminating "t", we get the trajectory: $t = x / 965.926$,

$$y = \frac{258.819x}{965.926} - \frac{9.81}{2}\left(\frac{x}{965.926}\right)^2$$

which is a <u>parabola</u>.

Ex: 3 A cannon is set to hit a target as shown Fig. 5. If the muzzle velocity is 750 m/s, at what angle α must the cannon be set to hit the target?

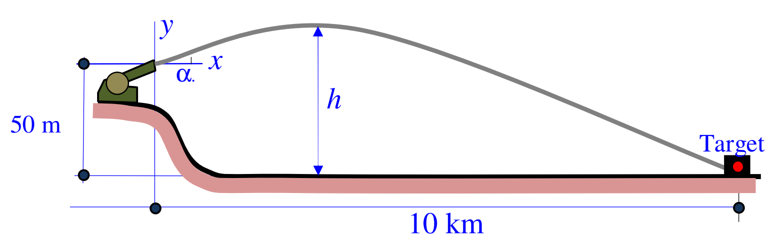

Figure 5

The acceleration components can be written as

$$\ddot{y} = -9.81t\ ;\ \ddot{x} = 0.$$

Integrating the above, we obtain

$$V_y = \dot{y} = -9.81t + A; V_x = \dot{x} = B,$$

from which we have

$$V_y(0) = A = 750\sin\alpha \text{ and } V_x(0) = B = 750\cos\alpha.$$

Integrating once again, we get

$$y = \frac{-9.81}{2}t^2 + 750\sin\alpha\, t + C \qquad \text{[A]}$$

$$x = 750\cos\alpha\, t + D \qquad \text{[B]}$$

At $t = 0$, $x = y = 0$, leads to $C = D = 0$.
From [B], we get

$$t = \frac{x}{750\cos\alpha}\ ,$$

and from [A]

$$y = \frac{-9.81}{2}\frac{x^2}{750^2\cos^2\alpha} + \frac{750\sin\alpha\cdot x}{750\cos\alpha}.$$

Using the condition $x = 10^4$ m, $y = -50$ m and $\sec^2\alpha = 1 + \tan^2\alpha$ in the above, we obtain

$$-50 = \frac{-9.81}{2}\frac{10^8}{750^2}(1+\tan^2\alpha) + \tan\alpha\, 10^4\,,$$

or $-872\tan^2\alpha + 10^4\tan\alpha - 872 = 0$; from which we get

$$\therefore \tan\alpha = \frac{-10^4 \pm\sqrt{10^8 - 4(872)(822)}}{-2\times 872} = 11.385, 0.0828.$$

The solutions of the above are: $\alpha = 84.98^o; 4.733^o$. The two possible firing angles are depicted in Fig. 6.

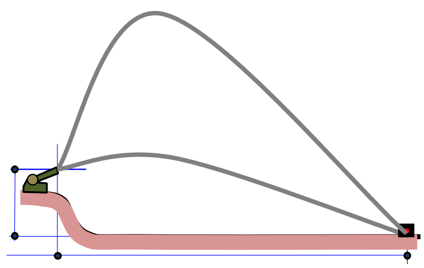

Figure 6

Ex: 4 The position of a particle at times $t = 5$ s, 2 s and 1 s are known to be; $\mathbf{r}(5) = (10\boldsymbol{i} + 20\boldsymbol{j} + 5\boldsymbol{k})$ m, $\mathbf{r}(2) = (5\boldsymbol{i} - 10\boldsymbol{j} + 3\boldsymbol{k})$ m and $\mathbf{r}(1) = (7\boldsymbol{i} + 5\boldsymbol{j} + 8\boldsymbol{k})$ m respectively. Determine the acceleration of the particle at $t = 4$s if the acceleration vector has the form, $\mathbf{a}(t) = At^2\,\boldsymbol{i} + Bt\,\boldsymbol{j} + Ct^3\,\boldsymbol{k}$ m/s^2; where A, B and C are constants.

$$V_x(t) = \frac{At^3}{3} + D \quad\text{and}\quad x(t) = \frac{At^4}{12}Dt + F.$$

Find A, D and F from the given conditions. Similarly,

$$V_y(t) = \frac{Bt^2}{2} + G \quad\text{and}\quad y(t) = \frac{Bt^3}{6} + Gt + H.$$

The remaining part is left as exercise.

Velocity and Acceleration in Term of Path Variables

At times, it is convenient to make use of certain geometrical parameters of the path.
We have seen earlier that

$$V = \frac{ds}{dt}\boldsymbol{\varepsilon}_t.$$

where ds/dt is the speed, and $\boldsymbol{\varepsilon}_t$ is the unit vector tangential to the path. The acceleration is obtained from the above by differentiating with respect to time as

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{d^2 s}{dt^2}\boldsymbol{\varepsilon}_t + \frac{ds}{dt}\frac{d\boldsymbol{\varepsilon}_t}{dt}.$$

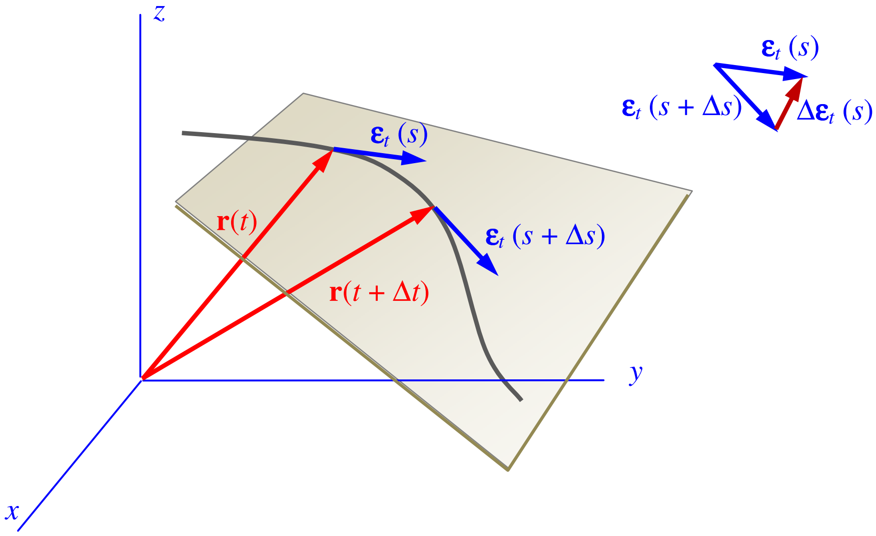

Figure 7

Now,

$$\frac{d\boldsymbol{\varepsilon}_t}{dt} = \frac{d\boldsymbol{\varepsilon}_t}{ds}\frac{ds}{dt}.$$

Therefore, we have

$$\mathbf{a} = \frac{d^2 s}{dt^2}\boldsymbol{\varepsilon}_t + \left(\frac{ds}{dt}\right)^2 \frac{d\boldsymbol{\varepsilon}_t}{ds}.$$

The derivative of the unit tangential vector $\boldsymbol{\varepsilon}_t$ with respect to s is obtained as follows. Consider $\boldsymbol{\varepsilon}_t$ at two Δs apart. The plane containing these two unit vectors, $\boldsymbol{\varepsilon}_t\ (s)$ and $\boldsymbol{\varepsilon}_t\ (s + \Delta s)$, is called the *osculating plane**, and is shown in Fig. 7. Therefore, it follows that

$$\frac{d\boldsymbol{\varepsilon}_t}{ds} = \lim_{\Delta s \to 0} \frac{\boldsymbol{\varepsilon}_t(s+\Delta s) - \boldsymbol{\varepsilon}_t(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta \boldsymbol{\varepsilon}_t}{\Delta s}.$$

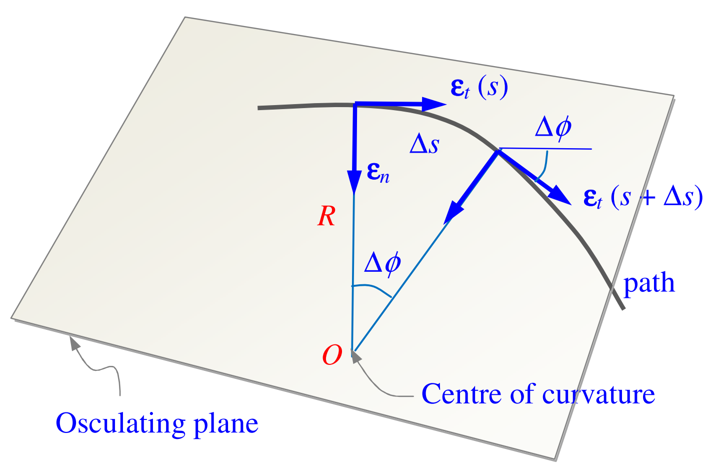

Figure 8

The limiting vector $\Delta\boldsymbol{\varepsilon}_t$ is denoted by $\boldsymbol{\varepsilon}_n$, and is called the *principal normal vector*, and is directed towards the centre of curvature as shown in Fig. 8.

* To *osculate* means to kiss. The plane is tangential to the curve (as if it kisses it).

$$\left|\Delta\boldsymbol{\varepsilon}_t\right| \approx \left|\boldsymbol{\varepsilon}_t\right|\Delta\phi = \Delta\phi$$

and $R\,\Delta\phi = \Delta s$. Therefore, we have

$$\left|\frac{\Delta\boldsymbol{\varepsilon}_t}{\Delta s}\right| = \frac{\Delta s}{R}\frac{1}{\Delta s} = \frac{1}{R}.$$

Hence

$$\frac{d\boldsymbol{\varepsilon}_t}{ds} = \frac{1}{R}\boldsymbol{\varepsilon}_n.$$

Therefore, the acceleration vector in terms of the path variables can be written as

$$\mathbf{a} = \frac{d^2 s}{dt^2}\boldsymbol{\varepsilon}_t + \frac{\left(\frac{ds}{dt}\right)^2}{R}\boldsymbol{\varepsilon}_n.$$

In the case of a three-dimensional curve, first get $\boldsymbol{\varepsilon}_t$ as a function of s. Differentiate $\boldsymbol{\varepsilon}_t$ with respect to s. Then get both R and $\boldsymbol{\varepsilon}_n$ (as the magnitude of $d\boldsymbol{\varepsilon}_t/ds = 1/R$, and the direction is along $\boldsymbol{\varepsilon}_n$. The unit vector normal to the osculating plane is called the *binormal vector*, and is given by $\boldsymbol{\varepsilon}_b = \boldsymbol{\varepsilon}_n \times \boldsymbol{\varepsilon}_t$.*

Ex: 5 A particle p moves along a circular path of radius 1m as shown in Fig. 9. As it crosses the y-axis it has an acceleration along the path of 2 m/s^2, and has a speed of 7 m/s in the negative x-direction. Determine the total acceleration of the particle.

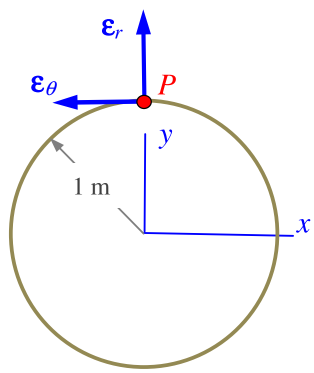

Figure 9

The total acceleration of the particle can be obtained as

$$\mathbf{a} = \frac{d^2 s}{dt^2}\boldsymbol{\varepsilon}_t + \frac{\left(\frac{ds}{dt}\right)^2}{R}\boldsymbol{\varepsilon}_n = 2(-\mathbf{i}) + \frac{7^2}{1}(\mathbf{j}) = (-2\mathbf{i} - 49\mathbf{j})\text{m/s}^2.$$

Ex: 6 Determine the unit vectors $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\varepsilon}_n$, and the radius of curvature at the point "a" in Fig. 10.

From the equation of the curve, it follows that

$$\frac{dy}{dx} = \frac{10}{x};\ \frac{d^2 y}{dx^2} = -\frac{10}{x^2}.$$

At "a", $y = 0$ and $x = 1$. Therefore

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* The Serret-Frenet's formulae are given by

$$\frac{d\boldsymbol{\varepsilon}_t}{ds} = \kappa\boldsymbol{\varepsilon}_n;\ \frac{d\boldsymbol{\varepsilon}_n}{ds} = -\kappa\boldsymbol{\varepsilon}_t + \tau\boldsymbol{\varepsilon}_b;\ \text{and}\ \frac{d\boldsymbol{\varepsilon}_b}{ds} = -\tau\boldsymbol{\varepsilon}_n,$$

where τ is known as the *torsion* of the curve and κ is the *curvature*. The curvature (which is the reciprocal of R) is a measure of rate of change of tangent vector, and torsion is a measure of rate of change of the binormal vector.

$$\frac{1}{R}=\frac{\left(\frac{d^2y}{dx^2}\right)}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}=\frac{10}{(1+100)^{3/2}}.$$

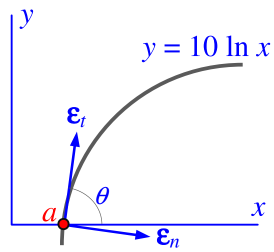

Figure 10

From the above, we get R = 101.504 m. Now,

$$\tan\theta=\frac{dy}{dx}=10\ \therefore\theta=84.29^o.$$

Therefore, we get

$$\boldsymbol{\varepsilon}_t=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}=0.0995\,\mathbf{i}+0.995\,\mathbf{j},\ \boldsymbol{\varepsilon}_n=\sin\theta\,\mathbf{i}-\cos\theta\,\mathbf{j}=0.995\,\mathbf{i}-0.0995\,\mathbf{j}\text{ and }\boldsymbol{\varepsilon}_b=\boldsymbol{\varepsilon}_n\times\boldsymbol{\varepsilon}_t=\mathbf{k}.$$

Ex: 7 The path of a particle is given by, $x(\tau)=A\sin\omega\tau$, $y(\tau)=A\cos\omega\tau$ and $z(\tau)=C\tau$, where A, C and ω are known constants. When the particle is at the x y plane (i.e. z = 0), it has a speed of V_o m/s and a rate of change of speed of N m/s^2. What is the acceleration of the particle at this position?

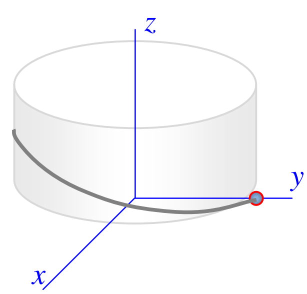

Figure 11

The unit tangent vector is obtained as

$$\boldsymbol{\varepsilon}_t=\frac{d\mathbf{r}}{ds}=\frac{d}{d\tau}\left(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}\right)\frac{d\tau}{ds}.$$

Now, as

$$\frac{dx}{d\tau}=A\omega\cos\omega\tau;\frac{dy}{d\tau}=-A\omega\sin\omega\tau;\frac{dz}{d\tau}=C\text{ and }ds=\sqrt{dx^2+dy^2+dz^2},$$

we get

$$ds=\sqrt{A^2\omega^2+C^2}\,d\tau.$$

Hence,

$$\boldsymbol{\varepsilon}_t=\frac{A\omega\left(\cos\omega\tau\,\mathbf{i}-\sin\omega\tau\,\mathbf{j}\right)+C\,\mathbf{k}}{\sqrt{A^2\omega^2+C^2}}.$$

The unit principal normal vector can be obtained as

$$\boldsymbol{\varepsilon}_n = R\frac{d\boldsymbol{\varepsilon}_t}{ds} = R\frac{d\boldsymbol{\varepsilon}_t}{d\tau}\frac{d\tau}{ds} = \frac{R}{\sqrt{A^2\omega^2 + C^2}}\frac{d\boldsymbol{\varepsilon}_t}{d\tau} = \frac{-RA\omega^2(\sin\omega\tau\,\mathbf{i} + \cos\omega\tau\,\mathbf{j})}{(A^2\omega^2 + C^2)}.$$

As $|\boldsymbol{\varepsilon}_n| = 1$, we have

$$R = \frac{A^2\omega^2 + C^2}{A\omega^2}.$$

At the position when the particle is on the xy-plane, we have $\tau = 0$. Then,

$$\boldsymbol{\varepsilon}_t = \frac{A\omega\mathbf{i} + C\mathbf{k}}{\sqrt{A^2\omega^2 + C^2}} \text{ and } \boldsymbol{\varepsilon}_n = \frac{-RA\omega^2\mathbf{j}}{A^2\omega^2 + C^2} = -\mathbf{j}.$$

$$\mathbf{a} = \frac{d^2s}{dt^2}\boldsymbol{\varepsilon}_t + \frac{\left(\frac{ds}{dt}\right)^2}{R}\boldsymbol{\varepsilon}_n = \frac{N}{\sqrt{A^2\omega^2 + C^2}}(A\omega\mathbf{i} + C\mathbf{k}) - \frac{V_0^{\ 2}}{R}\mathbf{j},$$

where $R = \dfrac{A^2\omega^2 + C^2}{A\omega^2}$.

CYLINDRICAL CO-ORDINATES

The position of P is denoted by means of R, θ and z coordinates. (z along the *axial* direction). $X = R\cos\theta$, $y = R\sin\theta$, $R = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$. The unit vectors $\boldsymbol{\varepsilon}_z$ (= $\mathbf{k}$), $\boldsymbol{\varepsilon}_R$, and $\boldsymbol{\varepsilon}_\theta$ are directed along the axial, radial and circumferential (or transverse) directions respectively. The circumferential direction is normal to the plane formed by $\boldsymbol{\varepsilon}_R$ and $\boldsymbol{\varepsilon}_z$, and has a sense given by right hand screw rule (with z, R, θ being the order of this permutation). See Fig. 12.

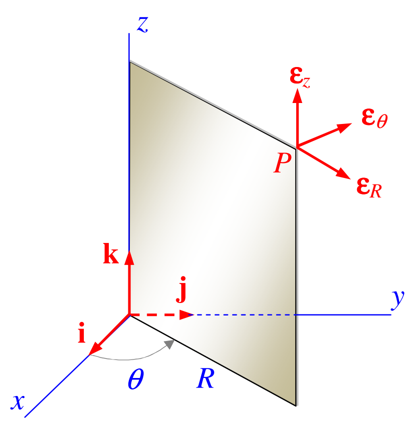

Figure 12

Thus, $\boldsymbol{\varepsilon}_z$ is a constant vector; $\boldsymbol{\varepsilon}_R$ and $\boldsymbol{\varepsilon}_\theta$ will change the direction as the particle moves. The components along the cylindrical coordinates are most apt for several problems: e.g. in turbo machine studies z-axis is the axis of rotation.
The position vector of P can be written as

$$\mathbf{r} = R\boldsymbol{\varepsilon}_R + Z\boldsymbol{\varepsilon}_z.$$

The velocity is obtained by differentiating the above with respect to t to get

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} = R\dot{\boldsymbol{\varepsilon}}_R + \dot{R}\boldsymbol{\varepsilon}_r + z\boldsymbol{\varepsilon}_z.$$

Evaluation of $\dot{\boldsymbol{\varepsilon}}_R$ is as follows:

$$\dot{\boldsymbol{\varepsilon}}_R = \frac{d\boldsymbol{\varepsilon}_R}{dt} = \frac{d\boldsymbol{\varepsilon}_R}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\boldsymbol{\varepsilon}_R}{d\theta}.$$

For a given R and z positions corresponding to θ and $\theta + \Delta\theta$, consider $\boldsymbol{\varepsilon}_R$ and its change $\Delta\boldsymbol{\varepsilon}_R$. Then

$$|\Delta\boldsymbol{\varepsilon}_R| = |\boldsymbol{\varepsilon}_R|\Delta\theta = \Delta\theta.$$

As $\Delta\theta \to 0$, the direction of $\Delta\boldsymbol{\varepsilon}_R$ approaches that of $\boldsymbol{\varepsilon}_\theta$. Thus

$$\Delta\boldsymbol{\varepsilon}_\mathrm{R} = |\Delta\boldsymbol{\varepsilon}_R|\boldsymbol{\varepsilon}_\theta = \Delta\theta\ \boldsymbol{\varepsilon}_\theta.$$

Therefore, we have

$$\frac{d\boldsymbol{\varepsilon}_R}{dt} = \dot{\theta}\ \boldsymbol{\varepsilon}_\theta.$$

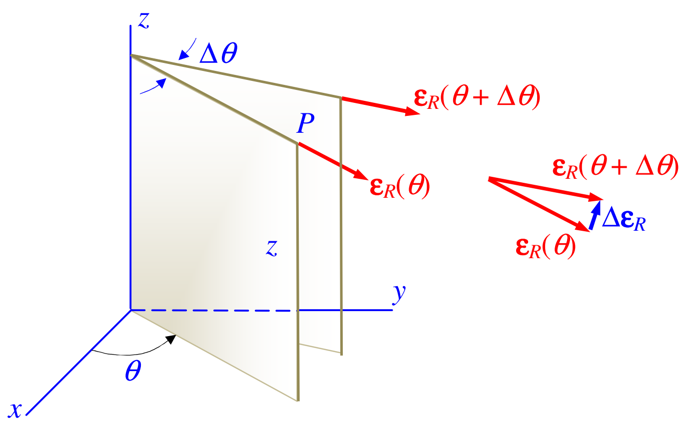

Figure 13

Thus, the velocity vector is obtained as

$$\mathbf{V} = R\dot{\theta}\boldsymbol{\varepsilon}_\theta + \dot{R}\boldsymbol{\varepsilon}_R + \dot{z}\boldsymbol{\varepsilon}_z.$$

Now, to get the acceleration of the particle, we proceed as follows:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \dot{R}\dot{\theta}\boldsymbol{\varepsilon}_\theta + R\ddot{\theta}\boldsymbol{\varepsilon}_\theta + R\dot{\theta}\dot{\boldsymbol{\varepsilon}}_\theta + \ddot{R}\boldsymbol{\varepsilon}_R + \dot{R}\dot{\boldsymbol{\varepsilon}}_R + \ddot{z}\boldsymbol{\varepsilon}_z.$$

We need to evaluation of $\dot{\boldsymbol{\varepsilon}}_\theta$ which is done as given below. Firstly, we can write

$$\dot{\boldsymbol{\varepsilon}}_\theta = \frac{d\boldsymbol{\varepsilon}_\theta}{dt} = \frac{d\boldsymbol{\varepsilon}_\theta}{d\theta}\dot{\theta}.$$

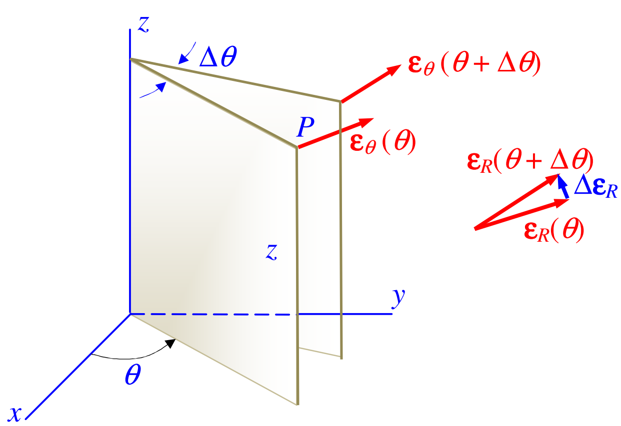

Figure 14

As $\Delta\theta \to 0$, the direction of $\Delta\boldsymbol{\varepsilon}_\theta$ approaches that of $-\boldsymbol{\varepsilon}_R$ as shown in Fig. 14. Hence, we get

$$\dot{\boldsymbol{\varepsilon}}_\theta = -\dot{\theta}\,\boldsymbol{\varepsilon}\,\mathbf{r}.$$

Thus, the acceleration vector can be obtained as

$$\mathbf{a} = \dot{R}\dot{\theta}\boldsymbol{\varepsilon}_\theta + R\ddot{\theta}\boldsymbol{\varepsilon}_\theta - R\dot{\theta}^2\,\boldsymbol{\varepsilon}_r.\ \ddot{R}\,\boldsymbol{\varepsilon}_R + \dot{R}\dot{\theta}\boldsymbol{\varepsilon}_\theta + \ddot{z}\,\boldsymbol{\varepsilon}_z.$$

Rearranging the above, we get

$$\mathbf{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\boldsymbol{\varepsilon}_r + \left(R\ddot{\theta} + z\dot{R}\dot{\theta}\right)\boldsymbol{\varepsilon}_\theta + \ddot{z}\,\boldsymbol{\varepsilon}_z.$$

For motion along a circular path in the *xy*-plane, we have $\dot{R} = \dot{z} = 0$ and $z = 0$. Thus

$$\mathbf{V} = r\dot{\theta}\boldsymbol{\varepsilon}_\theta \text{ and } \mathbf{a} = r\ddot{\theta}\boldsymbol{\varepsilon}_\theta - r\dot{\theta}^2\,\boldsymbol{\varepsilon}_r\,.$$

Also, $\boldsymbol{\varepsilon}_\theta$ is tangential to the path as shown in Fig. 15.

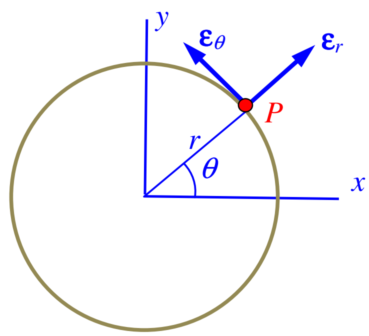

Figure 15

$\boxed{\text{Ex: } 8}$ The arm *OA* shown in Fig. 16 is 2m long, and its rotation about *O* is given by $\theta = 0.2t^2$ radians. The collar B slides along the arm such that its distance from *O* is given by $r = (2 - 0.18t^2)$ m. After the arm has rotated by 45°, determine (a) the total velocity of the collar, (b) the total acceleration of the collar, and (c) the relative acceleration of the collar with respect to the arm.

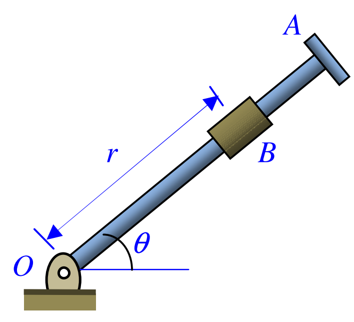

Figure 16

$$\theta = 0.2t^2\ ;\ \frac{\pi}{4} = 0.2t^2, \therefore t\,(\text{when } \theta = 45^\circ) = 1.98166\,\text{s}\,.$$

Calculate **V** and **a** from

$$\mathbf{V} = r\theta\boldsymbol{\varepsilon}_\theta + \dot{r}\,\boldsymbol{\varepsilon}_r,\ \ \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{\varepsilon}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\varepsilon}_\theta$$

Simple Kinematical Relations and Applications
Simple Relative Motion

It may be advantageous to employ two or more references in describing the motion of a particle. Consider two references *XYZ* (fixed or inertial) and *xyz* (moving) as shown in Fig. 18.

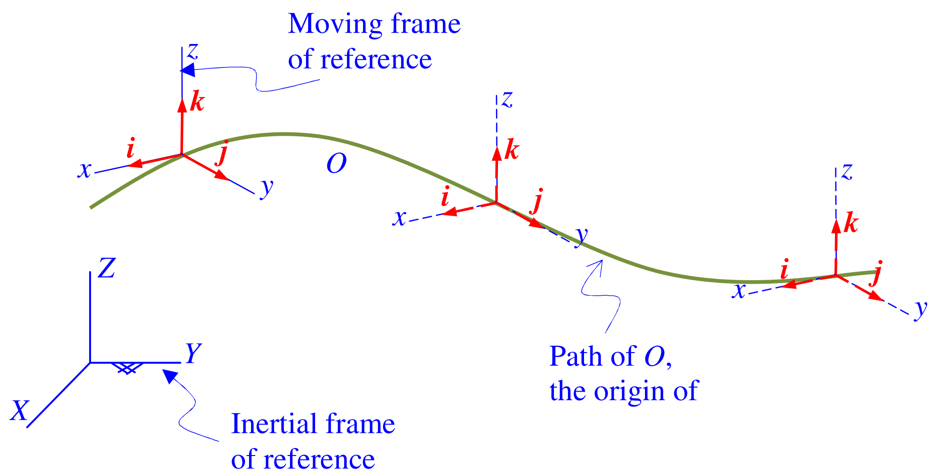

Figure 18

Let *xyz* translate (i.e. the direction of *xyz* axes always retain the same orientation.) with respect to *xyz*. Consider a vector $\boldsymbol{A}(t)$ which varies with time. In general, the time variation of $\boldsymbol{A}$ will depend on the reference system. Let

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} \text{ and } \left(\frac{d\mathbf{A}}{dt}\right)_{xyz}$$

represent the time derivatives of $\boldsymbol{A}$ as seen from *XYZ* and *xyz* respectively. Let us explore the relations between the above two. We have, working from first principles,

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \left[\frac{d}{dt}\left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right)\right]_{XYZ},$$

where A_x, A_y, A_z are the scalar components of $\boldsymbol{A}$ along the *xyz* axes.

As *xyz* translates relative to *XYZ*, the unit vector of *xyz*, viz. **i**, **j** and **k** are constant vectors as seen from *XYZ*. That is, although the lines of actions of **i**, **j**, **k** may change, their magnitude and direction as seen from *XYZ* are constant vectors. Therefore, we have

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \left(\frac{dA_x}{dt}\right)_{XYZ}\mathbf{i} + \left(\frac{dA_y}{dt}\right)_{XYZ}\mathbf{j} + \left(\frac{dA_z}{dt}\right)_{XYZ}\mathbf{k}.$$

In the above, A_x, A_y and A_z are scalar components of **A**. Their time derivative will not depend on the reference of observation. Thus

$$\left(\frac{dA_x}{dt}\right)_{XYZ} \equiv \left(\frac{dA_x}{dt}\right)_{xyz}$$

and hence can be indicated simply as $\frac{dA_x}{dt}$. Similarly, we have

$$\left(\frac{dA_y}{dt}\right)_{XYZ} \equiv \left(\frac{dA_y}{dt}\right)_{xyz} \equiv \frac{dA_y}{dt} \text{ and } \left(\frac{dA_z}{dt}\right)_{XYZ} \equiv \left(\frac{dA_z}{dt}\right)_{xyz} \equiv \frac{dA_z}{dt}.$$

Hence, we can write

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \frac{dA_x}{dt}\mathbf{i} + \frac{dA_y}{dt}\mathbf{j} + \frac{dA_z}{dt}\mathbf{k}\,.$$

Therefore, it follows that

$$\left(\frac{d\mathbf{A}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{A}}{dt}\right)_{xyz} \quad \text{and} \quad \left(\frac{d}{dt}\right)_{XYZ} = \left(\frac{d}{dt}\right)_{xyz}\,.$$

That is, the time derivative of a vector is the same for all reference axes that are *translating relative to one another*.

Note: In the above, we have made use of the fact that the unit vectors **i**, **j**, **k** of *xyz* relative to *XYZ* are constants. If *xyz rotates* relative to *XYZ*, this is not valid, and more complex relation becomes necessary.

A rigid body is said to translate if all its points are subject to the same velocity vector at every time *t*.

Motion of Particle Relative to a Pair of Translating Axes

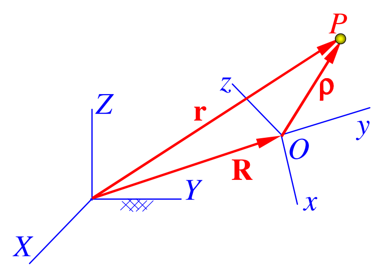

Figure 19

Consider a particle *P*, and a pair of references *XYZ* (inertial) and *xyz* (moving). The velocity **V** of *P* relative to *XYZ* is the time rate of change of position vector **r** for this reference. That is

$$\mathbf{V}_{XYZ} = \left(\frac{d\mathbf{r}}{dt}\right)_{XYZ}\,.$$

Similarly, the velocity with respect to *xyz* is given by

$$\mathbf{V}_{xyz} = \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz}\,.$$

Moreover,

$$\left(\frac{d\mathbf{R}}{dt}\right)_{XYZ}$$

is the velocity of the origin *O* of *xyz* reference as seen from *XYZ*. Since all points of *xyz* have the same velocity with respect to *XYZ*, the above is the velocity of reference *xyz* as seen from *XYZ*. Now, since, **r = R + P**, it follows that

$$\left(\frac{d\mathbf{r}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{R}}{dt}\right)_{XYZ} + \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ}$$

which can be rewritten as

$$\mathbf{V}_{XYZ} = \dot{\mathbf{R}} + \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{XYZ} = \dot{\mathbf{R}} + \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{xyz},$$

or

$$\mathbf{V}_{XYZ} = \dot{\mathbf{R}} + \mathbf{V}_{xyz}\,.$$

Similarly, the acceleration equation can be written as

$$\mathbf{a}_{XYZ} = \ddot{\mathbf{R}} + \mathbf{a}_{xyz},$$

where

$$\mathbf{a}_{XYZ} = \left(\frac{d\mathbf{V}_{XYZ}}{dt}\right)_{XYZ} \text{ and } \mathbf{a}_{xyz} = \left(\frac{d\mathbf{V}_{xyz}}{dt}\right)_{xyz}.$$

Thus, we can see from the above two equations that the motion of a particle relative to *XYZ* is equal to the motion of the particle relative to *xyz* plus the motion of *xyz* relative to *XYZ*. Note that these equations are valid only when *xyz* has a translatory motion as seen from *XYZ*.

Ex: 9 An aero plane is flying at a speed of 400 km/hr in a translatory manner relative to the ground reference *XYZ* as shown in Fig. 20. At the instant of interest, a downdraft causes the plane to accelerate downward at 30 km/hr/s. While this is happening, the pilot reduced the throttle so that the plane decelerates in the *Y*-direction at 20 km/hr/s. Thus, the acceleration of plane is $\mathbf{a} = -30\,\mathbf{k} - 20\,\mathbf{j}$ km/hr/s. While this is happening, a solenoid is operated to close a valve which weighs 2.5 N. What is the force on the valve gate from the plane at the instant when the valve gate is moving downward relative to plane at a speed of 2 m/s and accelerating downwards relative to plane at 5 m/s^2?

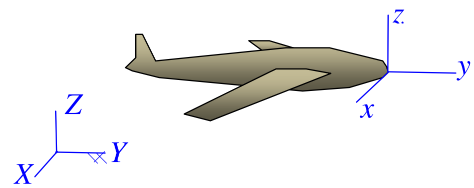

Figure 20

Motion of valve gate with respect to plane, $\mathbf{a}_{xyz} = -5\,\mathbf{k}$ m/s^2. The acceleration of *O*, the origin of *xyz*, with respect to *xyz*, $\ddot{\mathbf{R}} = -30\mathbf{k} - 20\mathbf{j}$ km/hr/s.

$$\mathbf{a}_{XYZ} = \mathbf{a}_{xyz} + \ddot{\mathbf{R}} = -5\,\mathbf{k} + (-30\,\mathbf{k} - 20\,\mathbf{j})\times 10^3/3600.$$

That is $\mathbf{a}_{XYZ} = -5.555\,\mathbf{j} - 13.333\,\mathbf{k}$ m/s^2. Now, From Newton's Law, we have: $\mathbf{F} = m\,\mathbf{a}_{XYZ}$. That is

$$\mathbf{F}_{\text{plane}} - 2.5\,\mathbf{k} = 2.5/9.81\times(-5.555\,\mathbf{j} - 13.333\,\mathbf{k}).$$

Therefore, $\mathbf{F}_{\text{plane}} = -1.4158\,\mathbf{j} - 0.898\,\mathbf{k}$ N.

Ex: 10 A ferries wheel rotates a the instant of interest with an angular speed of 0.5 rad/s and an angular acceleration of 0.1 rad/s^2 as depicted in Fig. 21a. A ball is thrown from the ground to an occupant a seat *A*. The ball arrives at the instant of interest with a velocity relative to the ground of $\mathbf{V}_{XYZ} = -10\,\mathbf{j} - 2\,\mathbf{k}$ ft/s. What is the velocity and the acceleration of the ball relative to the occupant at seat *A* provided the seat does not swing? (Exercise: 11.78 in the Book)

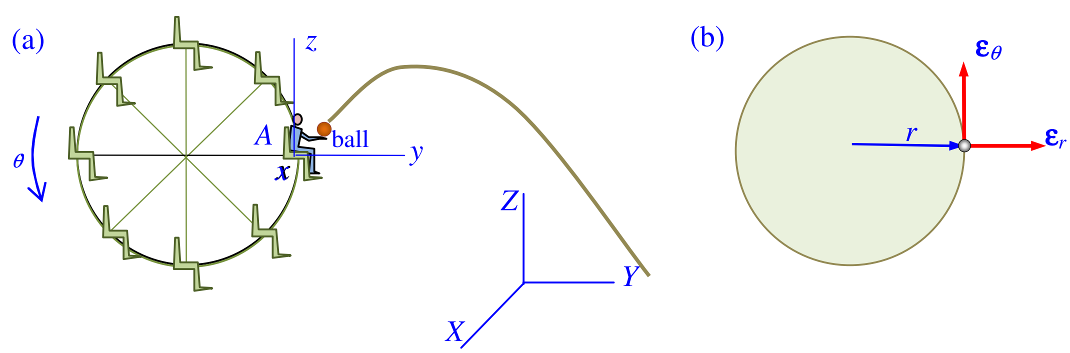

Figure 21

The radius of the wheel is 20 ft. The velocity of the ball can be written as

$$\mathbf{V}_{XYZ} = \mathbf{V}_{xyz} + \dot{\mathbf{R}}.$$

Velocity of the seat A (see Fig. 21b)

$$\dot{\mathbf{R}} = r\dot{\theta}\,\boldsymbol{\varepsilon}_\theta = 20\times0.5 = 10\ \mathbf{k}\ \text{ft/s}^2.$$

Therefore

$$-10\mathbf{j} - 2\mathbf{k} = \mathbf{V}_{xyz} + 10\mathbf{k}\ ,$$

and hence

$$\mathbf{V}_{xyz} = -10\mathbf{j} - 12\mathbf{k}\ \ \text{ft/s}.$$

The acceleration relative to ground is given by

$$\mathbf{a}_{XYZ} = \ddot{\mathbf{R}} + \mathbf{a}_{xyz}\ .\ \dot{\theta} = 0.5\,\text{rad/s}\,.$$

The acceleration of the seat A is

$$\ddot{\mathbf{R}} = r\ddot{\theta}\,\boldsymbol{\varepsilon}_\theta - r\dot{\theta}^2\boldsymbol{\varepsilon}_r = 20\times0.1\ \mathbf{k} - 20\times0.5^2\ \mathbf{j} = 2\ \mathbf{k} - 5\ \mathbf{j}\ \text{ft/s}^2.$$

Therefore,

$$-\,g\mathbf{k} = 2\mathbf{k} - 5\mathbf{j} + \mathbf{a}_{xyz}\ \text{ft/s}^2,$$

from which we get

$$\mathbf{a}_{xyz} = -32.2\mathbf{k} - 2\mathbf{k} + 5\mathbf{j} = 5\mathbf{j} - 34.2\mathbf{k}\,.$$

Particle-Dynamics

We examined the geometry of motion—*kinematics*—in the last section. We shall now consider the Newton's laws for the three systems of coordinates, viz., rectangular, cylindrical, and path coordinates.

Caution regarding units: 1 kg mass when acted upon by a force of 1 N accelerates, relative to an inertial frame of reference at 1m/s^2.

Rectangular Coordinates: Rectilinear Translation

Newton's Law for Rectangular Coordinates

In rectangular coordinates, Newton's law is expressed as

$$\mathbf{F} = m\,\mathbf{a},$$

where $\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$ and $\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$. Hence, the above in component form can be represented as

$$F_x = m\,a_x = m\frac{dV_x}{dt} = m\frac{d^2x}{dt^2}\ ,\ F_y = m\,a_y = m\frac{dV_y}{dt} = m\frac{d^2y}{dt^2}\ \ \text{and}\ \ F_z = m\,a_z = m\frac{dV_z}{dt} = m\frac{d^2z}{dt^2}\ .$$

- If the motion is known relative to an inertial frame of reference, we can easily get the resultant force acting on the particle from the above examinations.
- The inverse of this—determination of the motion, given the resultant force **F**—is not so simple.
- Let us begin with rectilinear translation in which the resultant force **F** has the same direction and line of action at all times.

Rectilinear Translation

Consider a body of mass m, executing rectilinear motion along x-axis as shown in Fig. 22. Then the Newton's second law can be written as

$$F = m\frac{d^2x}{dt^2}.$$

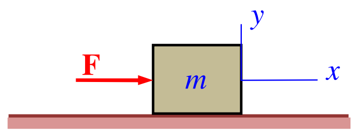

Figure 22

In the above, F can be

- a constant,
- a function of time t,
- a function of speed,
- a function of position, or
- a combination of the above.

1. Force is a function of time or a constant

Consider Fig. 22 again. The body of mass m is subjected to a force $F(t)$. The plane on which the body moves is assumed to be frictionless. The force of gravity is balanced by the normal reaction. Hence,

$$F(t) = m\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} = \frac{F(t)}{m}.$$

In the above,

- if $x\,(t)$ is known, we can easily find $F\,(t)$.
- if $F\,(t)$ is known, we need to integrate to obtain the motion. Thus

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{F(t)}{m}.$$

Integrating both sides, we get

$$\frac{dx}{dt} = V = \int \frac{F(t)}{m}dt + C_1.$$

Integrating the above once again yields

$$x(t) = \int\left[\int \frac{F(t)}{m}dt\right]dt + C_1 t + C_2,$$

where the constants of integration C_1 and C_2 are obtained from the initial conditions at $t = t_0$, $x = x_0$ and $V = V_0$. The above holds good even if $F\,(t)$ is a constant.

Ex: 11 A box weighing 500 N is sliding down an incline as shown. $\mu_d = 0.4$. (a) Find the time needed for the block to pick up a speed of 10 m/s. (b) What distance will the mass would have travelled by then?

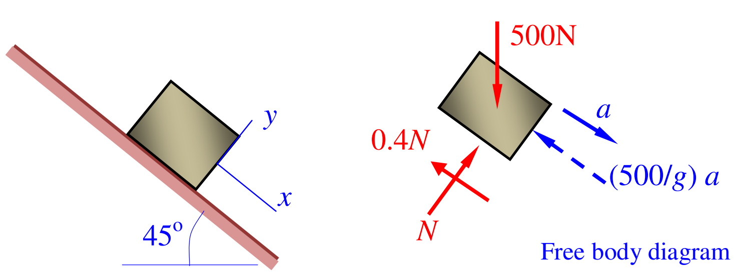

Figure 23

From the free body diagram shown in Fig. 23, we have N = 500 cos 45. Summing up forces along the x-direction, we obtain the equation of motion as

$$0.4\times\frac{500}{\sqrt{2}}-\frac{500}{\sqrt{2}}+\frac{500}{9.81}a=0,$$

from which we obtain $a=\ddot{x}=4.162\ \text{m/s}^2$. Integrating once, we get

$$\dot{x}=4.162t+C_1. \qquad [A]$$

The constant C_1 can be obtained using the initial condition that at t = 0, $\dot{x}=0$. Therefore, C_1 = 0. Integrating once again we get

$$\dot{x}=4.162\frac{t^2}{2}+C_2 \qquad [B]$$

The constant C_2 is obtained using the initial condition that at t = 0, x = 0. Therefore, C_2 = 0. The time t needed to reach a velocity of 10 m/s can be calculated from [A] as t = 10/4.162 = <u>2.403</u> s. The corresponding distance can be obtained from [B] as $x(t)$ = <u>12.013</u> m.

Ex: 12 A rigid body of mass m is acted upon by a force $F = A\cos\omega t$. Determine x and V if $x(t = 0) = V(t = 0) = 0$.

We have $m\ddot{x}=A\cos\omega t$. Integrating once, we get $\dot{x}=\frac{A}{m\omega}\sin\omega t+C_1$.

Integrating once again yields

$$x=-\frac{A}{m\omega^2}\cos\omega t+C_1t+C_2.$$

Using the boundary conditions $x(t = 0) = V(t = 0) = 0$, we obtain $C_1 = 0$ and $C_2 = A/m\omega^2$. Thus, the solution is

$$x(t)=\frac{A}{m\omega^2}(1-\cos\omega t).$$

Note: Consider a particle of mass m acted on by force $\mathbf{F}$. The linear momentum is defined as $\mathbf{P} = m\mathbf{V}(t)$. Newton's second law: the rate of change of linear momentum of a particle equals the sum of forces acting on it. That is $d/dt\,(mV) = \mathbf{F}$.

2. Force as a function of speed

Examples are: the aerodynamic drag force on an aircraft, or hydrodynamic forces on a towed body under water. The Newton's law can be written as

$$m\frac{dV}{dt}=F(V).$$

The above can be rearranged into

$$\frac{dV}{F(V)}=\frac{1}{m}dt$$

which on integration yields

$$\int\frac{dV}{F(V)}=\frac{t}{m}+C_1,$$

which gives V as a function of t.

Ex: 13 A race car runs at a speed of 200 km/hr. The aerodynamic drag force can be approximated as 0.18 V^2, where V is the speed in m/s. If the weight of the car is 30 kN, determine the distance the car would have moved by the time the speed drops to 150 km/hr, if the engine is shut down.

1 km/hr = 1000/3600 = 0.2778 m/s. Hence, 200 km/hr = 55.556 m/s and 150 km/hr = 41.667 m/s. The mass of the car is $m = 30\times10^3/9.81 = 3058.1$ kg.

Newton's Law can be written as

$$\frac{dV}{dt} = \frac{F(V)}{m} = \frac{-0.18V^2}{m},$$

which on rearranging yields

$$\frac{dV}{V^2} = \frac{-0.18}{m}dt.$$

Integrating the above, we get

$$-\frac{1}{V} = -\frac{0.18}{m}t + C_1. \qquad \text{[A]}$$

Let $t = 0$ when $V = 55.556$ m/s. Therefore $C_1 = -1/55.5556$. Rewriting the above equation as

$$-\frac{dt}{dx} = \frac{-0.81}{m}t - \frac{1}{55.5556},$$

and rearranging it as

$$\frac{dt}{\frac{0.18}{m}t + \frac{1}{55.556}} = dx.$$

The above is of form $dt/(at + b)$, which on integrating leads to

$$\ln\left[\frac{0.18}{3058.1}t + \frac{1}{55.556}\right] = \frac{0.18x}{3058.1} + C_2.$$

As at $t = 0$, $x = 0$, we get $C_2 = \ln(1/55.556)$. Hence, we have

$$\ln\left[\left(\frac{0.18}{3058.1}t + \frac{1}{55.556}\right)\times 55.556\right] = \frac{0.18}{3058.1} + x. \qquad \text{[B]}$$

From [A], when $V = 14.667$ m/s, we get $t = 101.93$. And from [B] corresponding to this time, we get $x =$ 4887.558 m.

3. Force as a function of position

Example is a spring mass system shown in Fig. 24.

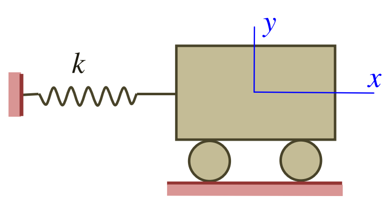

Figure 24

Newton's law can be written as

$$m\frac{dV}{dt} = F(x).$$

We cannot separate the variables here. However, we have

$$m\frac{dV}{dt} = m\frac{dV}{dx}\frac{dx}{dt} = mV\frac{dV}{dx}.$$

Therefore, we can write

$$mV\frac{dV}{dx} = F(x),$$

which can be rearranged and integrated to obtain

$$\frac{mV^2}{2} = \int F(x)\,d(x) + C_1.$$

$\boxed{\text{Ex: 14}}$ Determine $x(t)$ of the mass spring system shown in Fig. 24 if the mass is pulled to the right and released.

From Newton's law, we have

$$m\frac{dV}{dt} + kx = 0\ ,$$

which can be rearrange to get

$$mV\frac{dV}{dx} = -kx\,.$$

The above on integration yields

$$\frac{mV^2}{2} = \frac{-kx^2}{2} + C_1\,.$$

Alternatively, $m\ddot{x} + kx = 0$ is the equation of motion which is a homogeneous second order ordinary differential equation. Its general solution (which is same as the *complementary function*) is given by

$$x(t) = A\cos\omega t + B\sin\omega t\,.$$

Differentiating the above, we obtain the velocity as

$$V = \dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t\,.$$

Using eh initial conditions at $t = 0$, $x = x_0$, $V = V_0$, we get

$$A = x_0 \text{ and } B = V_0, \text{ where } \omega = (k/m)^{1/2}.$$

$\boxed{\text{Ex 15}}$ Consider the spring mass system shown in Fig. 25 with $M = 200$ kg and $k = 50$ N/mm. (a) If released slowly; what distance down the incline will the mass move to reach equilibrium? (b) If released suddenly, what speed will it acquire as it reaches the above equilibrium position?

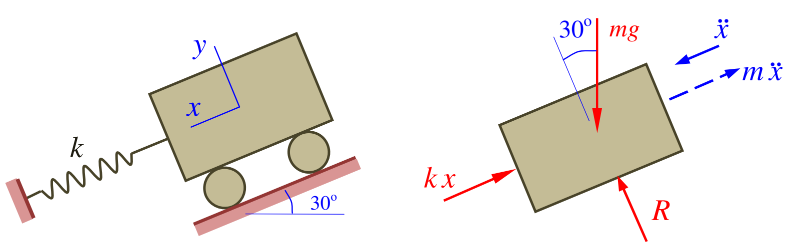

Figure 25

From Newton's law:

$$m\ddot{x} + kx = mg\sin 30 \text{ or } 200\ddot{x} + 5\times10^3 x = 200\times9.81\times0.5\,,$$

which can be rewritten as

$$\ddot{x} + 250x = 4.905\,.$$

The solution of the above second order differential equation is

$$x(t) = A\sin\omega t + B\cos\omega t + \frac{4.905}{250}\,,$$

which represents the displacement of the block. The last term of the above (= 0.01962 m) represents the static deflection (which is the *particular solution* of the given nonhomogeneous equation) which represents the answer to (a). The natural frequency of the system is $\omega = \sqrt{250}$.

The velocity of the block is obtained by differentiating the above to get

$$\dot{x} = A\omega \cos \omega t - B\omega \sin \omega t .$$

Using the initial conditions at $t = 0$, $x = 0$ and $\dot{x}=0$, we get $A = 0$ and $B = -0.01962$. Thus, the final solution is given by

$$x(t) = 0.01962(1 - \cos \omega t) .$$

Hence, the time needed to reach the static equilibrium position is

$$t\big|_{x=0.01962} = 0.1\,\text{s},$$

and the corresponding speed is

$$\dot{x}(0.1) = -0.01962\sqrt{250}\sin(\sqrt{250}\times 0.1) = \underline{0.3102}\ \text{m/s}.$$

NEWTON'S LAW FOR CYLINDRICAL COORDINATES

The Newton's law for cylindrical coordinates can be written as

$$F_R = m(\ddot{R} - R\dot{\theta}^2) ,$$

$$F_\theta = m(R\ddot{\theta} + 2\dot{R}\theta) ,$$

and

$$F_z = m\ddot{z} .$$

- If the motion is known, we can easily determine the force components.
- If the force components are known, it is extremely difficult to solve for the motion as the equations are nonlinear.

$\boxed{\text{Ex: 16}}$ A platform as shown in Fig. 26 has a constant angular velocity ω = 5 rad/s. A block B of mass 2 kg slides in a frictionless chute attached to the platform and is connected to the end of a very light spring of constant k = 20 N/m. When the block B is at r = 100mm, the spring is unstretched. If the mass B is released at r = 300mm from a stationary position relative to the platform, what is its speed relative to the platform when it has moved to r = 400 mm? What is the transverse force on the body B at this position?

From Newton's Laws along the radial direction, we have

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) = -k\left(r - 0.1\right).$$

That is $2\left(\ddot{r} - 25r\right) = -20\left(r - 0.1\right)$, which can be simplified further as $\ddot{r} = 15r + 1$. Now,

$$\ddot{r} = \frac{dV_r}{dt} = \frac{dV_r}{dr}\frac{dr}{dt} = V_r\frac{dV_r}{dr} = 15r + 1 .$$

Rearranging the above, we obtain

$$V_r\, dV_r = (15r + 1)dr ,$$

which on Integration yields

$$\frac{V_r^{\,2}}{2} = \frac{15r^2}{2} + r + C_1 .$$

Now, $V_r = 0$ at $r = 0.3$ m. Therefore, $C_1 = -0.975$. Hence we can write

$$V_r^{\,2} = 15r^2 + 2r - 1.95 .$$

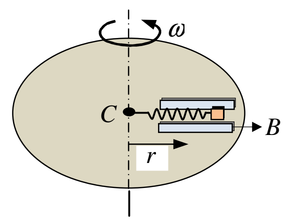

Figure 26

Consequently, when $r = 0.2$ m, V_r = 1.118 m/s. The corresponding transverse force is given by

$$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = \underline{22.361}\text{ N.}$$

Newton's Law for Path Variables

Newton’s law in terms of the path variables can be written as

$$F_t = m\frac{d^2 s}{dt^2}$$

and

$$F_n = m\frac{\left(\dfrac{ds}{dt}\right)^2}{R}.$$

Although the former is a linear differential equation, the latter is nonlinear.

$\boxed{\text{Ex: 16}}$ (as 12.120) A skier is moving down the hill at a speed of 30 ms/hr, when he is at the position shown in Fig. 27. If he weighs 180 lb, what total force does he exert on the snow surface? Assume that the coefficient of friction in 0.1. The hill is parabolic.

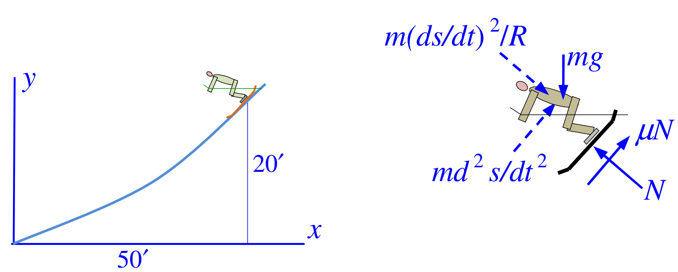

Figure 27

The equation of the parabola is: $y = a\,x^2$. As at $x = 50'$, $y = 20'$, $a = 0.008$. The derivatives are $y' = 2\,a\,x$, and $y'' = 2\,a$. Therefore, the curvature at the point is

$$\frac{1}{R} = \frac{y''}{\left(1 + (y')^2\right)^{3/2}} = 0.007618.$$

Now, since

$$\tan\theta = y';\ \cos\theta = 0.78086;\ \text{and}\ \frac{ds}{dt} = \frac{30\times 5280}{60^2}\,\text{ft/s},$$

we obtain (from the free body diagram shown in Fig. 27)

$$N = mg\cos\theta + \frac{m}{R}\left(\frac{ds}{dt}\right)^2 = 180\cos\theta + \frac{180}{32.2}\times\frac{1}{R}\left(\frac{ds}{dt}\right)^2 = 223.001\ \text{lb.}$$

And hence, the resultant force exerted by skier on the ground is obtained as

$$F_R = \sqrt{N^2 + (\mu N)^2} = \underline{224.112}\ \text{lb.}$$

Exercise 1: A block A of mass 20 kg rests on another block of mass 12 kg as shown in Fig. E1. A force of 150 N is applied on the block A as shown. If μ_d between A and B is 0.6 and that between B and the ground is 0.12, determine the relative speed of A with respect to B at t = 0.2s if the system starts from the rest.

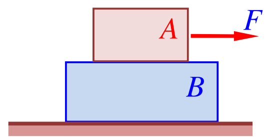

Figure E1

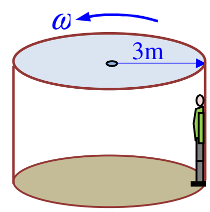

Figure E2

Exercise 2: A circular room revolves at ω when the floor is suddenly depressed as shown in Fig. E2. If the occupants standing close to the walls were to remain glued to the wall, what is the minimum angular velocity ω? μ_s = 0.4. [Answer: ω = 27.3 rpm = 2.859rad/s.]

Exercise 3: A conical pendulum of length 1 m is made to rotate at a constant angular speed of 5 rad/s, about the vertical axis as depicted in Fig. E3. Determine the tension in the chord if the mass of the pendulum bob in 0.5 kg. What is θ? [Answer: T = 12.5N, $\theta = 66.9^o$.]

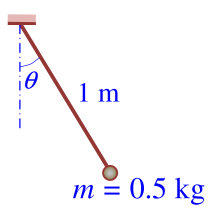

Figure E3

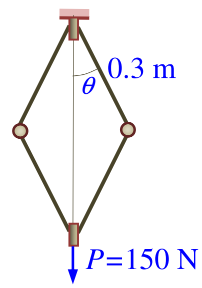

Figure E4

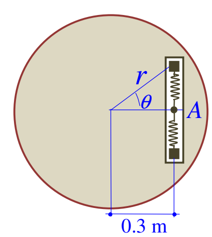

Figure E5

Exercise 4: A fly-ball governor B as shown in Fig. E4 is subjected to a pull of p = 150N. If the balls are each of weight 100N, what is the angular velocity required to maintain the configuration shown. In Fig. E4 (i.e., with $\theta = 30^o$.

Exercise 5: A horizontal platform rotates at a constant angular speed of 5 rad/s. Fixed to the platform is a friction less chute in which two identical masses each of 2 kg are constrained by a pair of identical linear springs of k = 250 N/m. If the unstretched length of each spring is 0.18 m, show that at steady state $\theta = 36.81^o$. The springs are fixed to the platform at A.

A System of Particles

The General Motion of a System of Particles

Consider a system of particles as depicted in Fig. 28 that have interactions between one another. The interaction forces obey Newton's 3rd law, (i.e. they are equal and opposite).

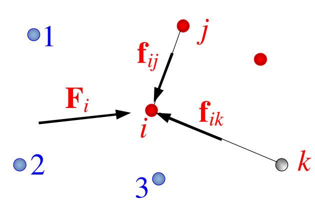

Figure 28

For particle "i", Newton's second law can be written as

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{f}_{ij} \,,$$

where $\mathbf{F}_i$ is the external force acting on i^{th} particle, $\mathbf{f}_{ij}$ is the force on particle i from particle j. Moreover, it follows from Newton's third law that $\mathbf{f}_{ij} = -\,\mathbf{f}_{ji}$. Writing the above equation for each of the particles, and summing up we obtain

$$\sum_{i=1}^{n} m_i \frac{d^2}{dt^2}\mathbf{r}_i = \sum_{i=1}^{n} F_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{f}_{ij} \,,$$

which can be simplified to obtain

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i = \frac{d^2}{dt^2}\left(\sum_{i=1}^{n} m_i\,\mathbf{r}_i\right).$$

The quantity $\Sigma_i\, m_i\, \mathbf{r}_i$ is defined as the first moment of mass of the system of particles. Thus, if $\mathbf{r}_c$ is the position vector of the *centre of mass*, as depicted in Fig. 29, then we have

$$M\,\mathbf{r}_c = \sum_{i=1}^{n} m_i \mathbf{r}_i \,,$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass of the system of particles.

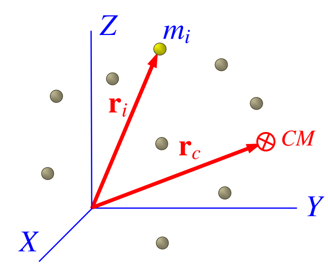

Figure 29

Thus, we can write the Newton's law for the system of particles as

$$\mathbf{F} = \frac{d^2}{dt^2} M\,\mathbf{r}_c = M\,\frac{d^2\mathbf{r}_c}{dt^2}\,.$$

In the above, $\mathbf{F}$ is the total force acting on all the particles. This equation implies that the system of particles could be considered as if it is a single particle of mass M situated at the centre of mass with position vector $\mathbf{r}_c$.

$\boxed{\text{Ex: 17}}$ An astronaut on a space walk pulls a man A of 150 kg towards him and shortens the distance d by 3 m. If the weight of the astronaut is 700 N on earth, how far does the mass A move from its original position? The cord may be treated as massless.

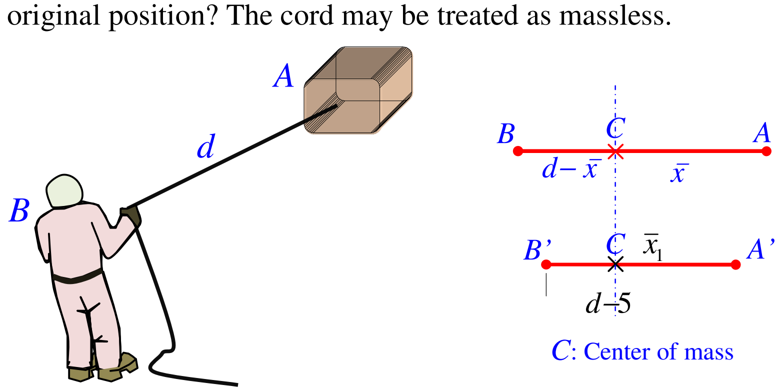

Figure 30

Taking moments of mass about A, we get

$$\left(150 + 700/g\right)\bar{x} = \left(700/g\right) d \, . \qquad \text{[A]}$$

Now, taking moments about A again, but in the new position, we obtain

$$\left(150 + 700/g\right)\bar{x}_1 = \left(700/g\right)(d - 3) \, . \qquad \text{[B]}$$

Subtracting [B] from [A] yields the answer as

$$\bar{x} - \bar{x}_1 = \frac{700 \times 3}{9.81\left(150 + 700/g\right)} = \underline{0.967} \text{ m.}$$

Exercise: 5 Three bodies have the weights 50 N, 20 N and 30 N respectively. Their positions at time t are given below:

$$\mathbf{r}_1 = \begin{Bmatrix} 5 \\ 3 \\ 2 \end{Bmatrix} \text{m}, \ \ \mathbf{r}_2 = \begin{Bmatrix} 4 \\ 0 \\ -4 \end{Bmatrix} \text{m, and } \mathbf{r}_3 = \begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix} \text{m.}$$

Determine the position of centre of mass at time t. What is the velocity of the centre of mass if the velocity of each body is:

$$\mathbf{V}_1 = \begin{Bmatrix} 3 \\ -2 \\ 0 \end{Bmatrix} \text{m/s}, \ \ \mathbf{V}_2 = \begin{Bmatrix} 0 \\ 5 \\ 1 \end{Bmatrix} \text{m/s, and } \mathbf{V}_3 = \begin{Bmatrix} 4 \\ 0 \\ 0 \end{Bmatrix} \text{m/s.}$$

If the following external forces act on the particles:

$$\mathbf{F}_1 = \begin{Bmatrix} 5 \\ 0 \\ 8 \end{Bmatrix} \text{N}, \ \ \mathbf{F}_2 = \mathbf{0} \text{, and } \mathbf{F}_3 = \begin{Bmatrix} 3t \\ 2 \\ -8 \end{Bmatrix} \text{N},$$

where t is the time in seconds. What is the acceleration of the centre of mass? What is its position after 5 sec from that given initially?
