ZZU102 ENGINEERING MECHANICS II—DYNAMICS

MODULE 2

Energy Methods for Particles

Analysis for a single particle

Certain problems can be easily solved by the method of energy. Consider Fig. 2.1 in which *XYZ* represents an inertial frame of reference.

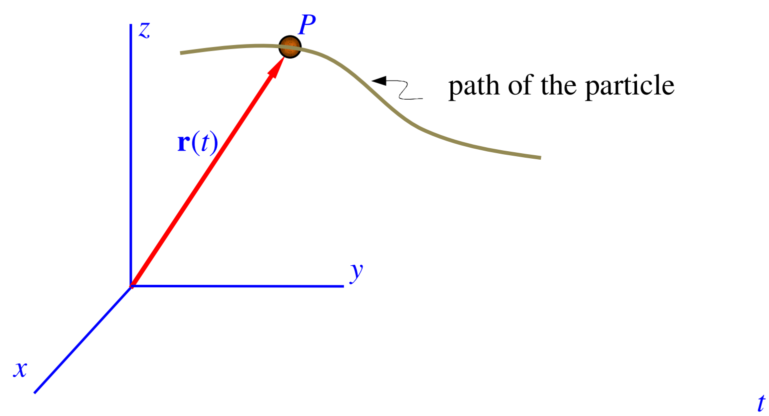

Figure 2.1

Newton's second law can be written as

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d\mathbf{V}}{dt}.$$

Take dot products on both sides of the above equation with $d\mathbf{r}$ and integrate from position 1 to 2 to get

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = m\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{d\mathbf{V}}{dt}\cdot d\mathbf{r} = m\int_{t_1}^{t_2} \frac{d\mathbf{V}}{dt}\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

Note the change of variable from $\mathbf{r}$ to t in the last term. We can rearrange the integrand of the last term as

$$\frac{d\mathbf{V}}{dt}\cdot\frac{d\mathbf{r}}{dt} = \mathbf{V}\cdot\frac{d\mathbf{V}}{dt} = \frac{1}{2}\frac{d}{dt}\left(\mathbf{V}\cdot\mathbf{V}\right) = \frac{1}{2}\frac{d}{dt}V^2.$$

Hence we obtain

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}m\int_{t_1}^{t_2}\frac{d}{dt}V^2 dt = \frac{1}{2}m\int_{v_1}^{v_2} d(V^2) = \frac{1}{2}mV_2^{\,2} - \frac{1}{2}mV_1^{\,2},$$

which can be rewritten as

$$\mathcal{W}_{1-2} = \frac{1}{2}mV_2^{\,2} - \frac{1}{2}mV_1^{\,2}.$$

In the above equation, the left hand side integral denoted by $\mathcal{W}_{1-2}$, is the work integral

$$\mathcal{W}_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \bullet d\mathbf{r}$$

and, in general, depends on the path between $\mathbf{r}_1$ and $\mathbf{r}_2$ and is known as a *path function*. However, the right hand side integral involves the kinetic energy (= ½ mV^2) depends on the instantaneous state of motion of the particle and is therefore a *point function*, and is dependent of the path.

The above equation is valid for,

- a single particle,
- a system of particles: (then, the centre of mass is relevant),
- a rigid body in translation, and
- a body whose size is small when compared to its trajectory. In such a case, however, the velocity and acceleration of the centre of mass may be quite different from those of the other points.

Now, consider a component of Newton's law in one direction, say the x-direction. Then, we can write

$$F_x \mathbf{i} = m \frac{dV_x}{dt} \mathbf{i} \, .$$

Taking dot product of the above equation with $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$, we obtain

$$\int_{x_1}^{x_2} F_x dx = \frac{m}{2}\left[(V_x)_2^{\,2} - (V_x)_1^{\,2}\right].$$

That is, the work done on a particle in any direction equals the change in kinetic energy associated with the component of velocity in that direction.

- It is advantageous to employ the energy principle when velocities are desired and forces are functions of positions.
- Any problem solved by Newton's law, can also be solved by the energy method.
- The choice of the method depends on convenience.

Ex: 2.1 A car is moving at the 60 km/hr when the driver jammed on his brakes. What distance will the car skid before stopping if the coefficient of dynamic friction, μ_d = 0.6 between the tyre and road? The weight of the car is 160 kN.

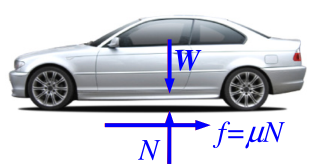

Figure 2.2

See Fig. 2.2. The frictional force is given by $f = \mu N = 0.6\times160\times10^3$ N. Equating the work done to the change in kinetic energy, we obtain

$$-0.6\times16\times10^3\times d = \frac{1}{2} m\, 0^2 - \frac{1}{2} m V_1^{\,2}$$

As $m = 16\times10^3/9.81$, we obtain the distance d = <u>23.6</u> m.

Ex: 2.2 If the system shown in Fig. 2.3 is released from rest, what is the speed at which W_2 hits the ground? W_1=250 N, W_2 = 200 N, and μ = 0.3.

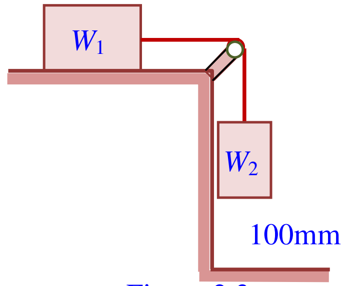

Figure 2.3

A. Direct solution using Newton's law:

Consider the two free body diagrams shown in Fig. 2.4. Using D'Alembert's concept of the inertia force, the (fictitious) inertia forces are marked by dotted arrows in the figure.

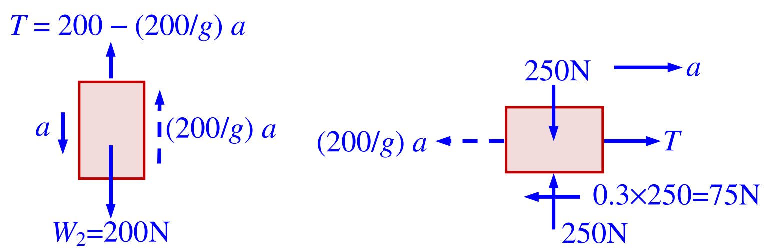

Figure 2.4

Consider the first free body diagram. Writing the dynamic equation of equilibrium, we get

$$T = 200 - \frac{200}{g} a .$$

From the second free body diagram, we again get

$$T = 75 + \frac{250}{g} a = 200 - \frac{200}{g} a ,$$

from which we get the acceleration as, a = <u>2.7242</u> m/s^2 and the tension in the cord as, T = <u>144.4</u> N. Integrating the acceleration, we obtain the velocity as

$$V = 2.7242 t + C ,$$

As at t = 0, V = 0, we have C = 0. Integrating the above once again, we get the displacement of block W_2 as

$$y = 2.7242\ t^2/2 + D$$

At t = 0, y = 0, we have D = 0. Time to reach ground: $y = 100\,\text{mm}$ = 0.1m, therefore, t = 0.2709s, and $V\,(t)$ = <u>0.738</u> m/s.

B. Using work-kinetic energy equation:

Equating the work done to the change in kinetic energy, we get

$$-0.3 \times 250 \times 0.1 + 200 \times 0.1 = \frac{1}{2}\left(\frac{250}{g} + \frac{200}{g}\right) V^2 ,$$

from which we get the velocity directly as V = <u>0.738</u> m/s.

Power Considerations

Power is the rate of performing work. That is

$$\text{Power} = \frac{dW}{dt} = \frac{\sum_{i=1}^{n} \mathbf{F}_i \bullet d\mathbf{r}_i}{dt} = \sum_{i=1}^{n} \mathbf{F}_i \bullet \mathbf{V}_i \,,$$

where $\mathbf{V}_i$ is the velocity of the point of application of the i^{th} force.

Ex: 2.3 A collar having a mass of 5 kg can slide without friction on a pipe as depicted in Fig. 2.5. If released from rest at the position shown where the spring is unstretched, what speed will the collar have after moving 50 mm? K = 2000 N/m.

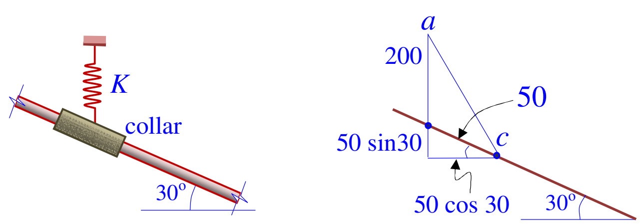

Figure 2.5

Stretch of the spring $ac - ab = 29.129$ mm. Writing the work-energy equation, we obtain

$$0 + 0 + \frac{50}{1000}\sin 30 \times 5 \times 9.81 - \frac{1}{2} \times 2000 \times x^2 = \frac{1}{2} \times 5 \times V^2 \,,$$

from which we obtain the solution as $V = \underline{0.3885}$ m/s.

Ex: 2.4 If the system shown in Fig. 2.6 is released from rest, with what speed will block B hit the ground?

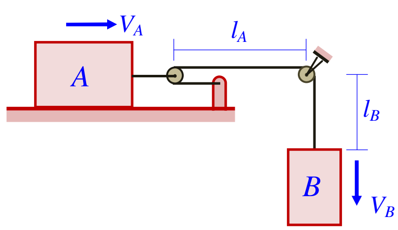

Figure 2.6

Since the length of the cord remains unchanged, we can write

$$2l_A + l_B = C \,.$$

Therefore, it follows that

$$2\dot{l}_A + \dot{l}_B = 0 \,,$$

from which we get the relationship between the velocities of the block.

$$\dot{l}_A = -V_A \text{ and } \dot{l}_B = V_B \,.$$

Using this relation and knowing other data, it is a straightforward exercise to complete this problem using the energy method.

Conservative Force Fields

Consider a body acted only by gravity force W as an active force and moving along a frictionless path from position 1 to position 2 as shown in Fig. 2.7. The work done by gravity can be written as

$$\mathcal{W}_{1-2} = \int_1^2 \mathbf{F} \bullet d\mathbf{r} = \int_1^2 (-W\mathbf{j}) \bullet d\mathbf{r} = -W \int_1^2 dy = W(y_1 - y_2).$$

The work done does not depend on the path, but depends only on the end points 1and 2. Force fields which depend only on end positions and are independent of the path are called *conservative force fields*.

For a conservative force field $\mathbf{F}(x, y, z)$, the work done as it moves from 1 to 2 can be written as

$$\mathcal{W}_{1-2} = \int_1^2 \mathbf{F} \bullet d\mathbf{r} = V_1(x, y, z) - V_2(x, y, z),$$

where V is a scalar function evaluated at the end points called *potential (energy) function*.

The change in potential ΔV is the negative of work done by this force field in going from position 1 to 2. For a closed path: $\oint \mathbf{F} \bullet d\mathbf{r} = 0$.

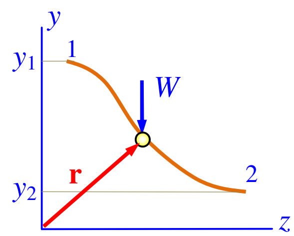

Figure 2.7

Consider an arbitrary infinitesimal path segment $d\mathbf{r}$ stating from 1. Then, for the above to hold good, we need $\mathbf{F} \bullet d\mathbf{r} = -dV$. This can be rewritten as

$$F_x\, dx + F_y\, d_y + F_z\, dz = -\left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right).$$

For the above equation to hold good, we have $F_i = -\partial V/\partial x_i$, $i = 1, 2, 3$, or $\mathbf{F} = -\, \mathbf{grad}\ V = -\nabla V$, where

$$\nabla(\ ) = \frac{\partial}{\partial x}(\ )\mathbf{i} + \frac{\partial}{\partial y}(\ )\mathbf{j} + \frac{\partial}{\partial z}(\ )\mathbf{k}$$

is the gradient operator.

Thus, a conservative force field must be a function of position, and must be expressible as the gradient of a scalar function. Or, if a force $\mathbf{F}$ is a function of position and can be expressed as the gradient of a scalar field, it must be a conservative force.

CONSTANT FORCE FIELD:

If the force field is constant at all positions, it can be represented as the gradient of the scalar function (the potential function) $V = -(ax + by + cz)$. Hence, we have $\mathbf{F} = -\nabla V = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. An example for this is the gravitational field $\mathbf{F} = -mg\mathbf{k}$. Then $V = mgz$.

FORCE PROPORTIONAL TO LINEAR DISPLACEMENTS:

If $\mathbf{F} = -K\,x\,\mathbf{i}$, then the potential function is $V = ½\,Kx^2$ so that $\mathbf{F} = -\nabla V$. The reader will recall that such a force field can be produced by a linear spring of having a spring constant K.

Conservation of Mechanical Energy

Consider the motion of a particle subjected to a conservative force field. We have seen that

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \bullet d\mathbf{r} = \frac{1}{2} m V_2^{\,2} - \frac{1}{2} m V_1^{\,2}.$$

Moreover, for a conservative field we can also write

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \bullet d\mathbf{r} = (\mathrm{PE})_1 - (\mathrm{PE})_2 ,$$

where $(\mathrm{PE})_i$ represents the potential energy function at the i-th stage (i = 1 or 2). From the above two equations, equating the right hand sides, we get

$$(\mathrm{PE})_1 - (\mathrm{PE})_2 = (\mathrm{KE})_2 - (\mathrm{KE})_1,$$

where (KE) is the kinetic energy of the particle. The above equation can be rearranged to yield

$$(\mathrm{PE})_1 + (\mathrm{KE})_1 = (\mathrm{PE})_2 + (\mathrm{KE})_2,$$

or the total energy = (PE) + (KE) = a constant. This is the *law of conservation of mechanical energy* for a conservative system.

Ex: 2.5 A particle is dropped from rest down a frictionless chute as shown in Fig. 2.8. What is the velocity as it falls down by h?

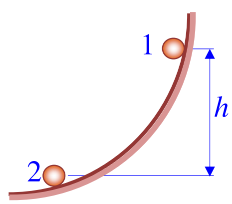

Figure 2.8

The work-energy equation can be written as

$$mgh + 0 = 0 + \frac{1}{2} m V^2 ,$$

from which we obtain the answer as $V = \sqrt{2gh}$.

Ex: 2.6 A block of mass m is dropped on to a spring as shown in Fig. 2.9. What is $\delta_{\max}$?

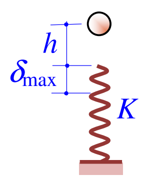

Figure 2.9

At $\delta_{\max}$, $V = 0$. Therefore, $mg(h + \delta_{\max}) + 0 = ½ K \delta_{\max}^2$. Solve this quadratic equation for $\delta_{\max}$.

Ex: 2.7 A vehicle of mass 1800kg is being transported in a rail rod car as shown in Fig. 2.10. The spring constant K= 20 kN/m. If the rail road car is stopped all of a sudden, what is the maximum spring force developed?

Writing the work-energy equation

$$\frac{1}{2} \times 1800 \times \left(\frac{5000}{3600} \right)^2 = \frac{1}{2} K x^2 ,$$

from which we get x = 0.4166 m. Therefore, the force in the spring = Kx = 8333.33 N

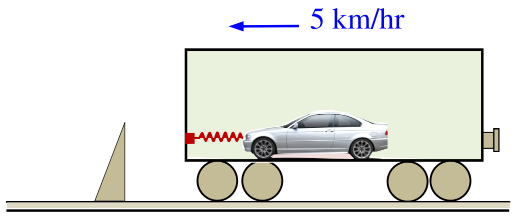

Figure 2.10

Ex: 2.8 In Fig. 2.11, the spring is nonlinear with spring force of $0.06x^2$. If the block of weight W = 225 N is suddenly released from rest, what is the maximum deflection of the spring?

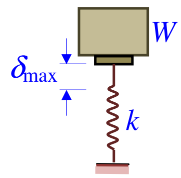

Figure 2.11

From work-energy equation, we have

$$-W\delta + \int_0^{\delta} 0.06x^2 dx = 0,$$

from which we obtain δ = 106.067 mm.

Ex: 2.9 Is the following force field conservative? (a) $\mathbf{F} = (10z + y)\mathbf{i} + (15yz + x)\mathbf{j} + \left(10x + \frac{15y^2}{2}\right)\mathbf{k}$.

If **F** = − ∇V, **curl F** = **0**. Thus, in order to check whether a given force field is conservative, we need to just check whether **curl F** = **0**.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 10z + y & 15yz + x & 10x + \frac{15y^2}{2} \end{vmatrix}$$

$$= \mathbf{i}(15y - 15y) + \mathbf{j}(10 - 10) + \mathbf{k}(1 - 1) = \mathbf{0}.$$

Therefore **F** is conservative.

Alternative Form of Work-Energy Equation

We shall see an alternative energy equation which resembles the first law of thermodynamics, and which has much physical appeal. Consider the case where certain of the forces acting on a particle are conservative while others are not. Recall that for conservative forces the negative of the change in potential energy between positions 1 and 2 equals the work done by these (conservative) forces as the particles goes from 1 to 2, along any path. We can thus write

$$\int_1^2 \mathbf{F} \bullet d\mathbf{r} - \Delta(\text{PE})_{1,2} = \Delta(\text{KE})_{1,2},$$

where the integral represents the work of the nonconservative forces; the operator Δ represents the final state minus the initial state. Calling this integral $\mathcal{W}_{1-2}$ we have:

$$\Delta(\mathrm{KE} + \mathrm{PE}) = \mathcal{W}_{1-2}.$$

Thus the work of the nonconservative forces goes into changing the total energy (= KE + PE) of the particle.

Ex: 2.10 When the 1000 N block of Fig. 2.12 slides down by 2m, what velocity will it require?

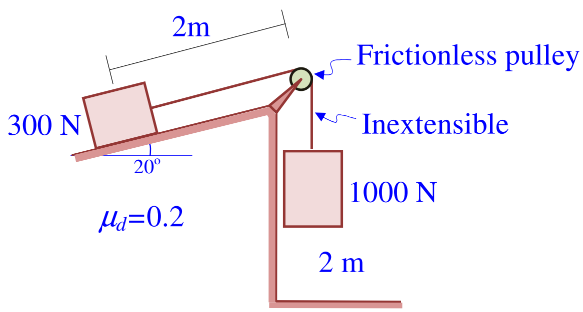

Figure 2.12

Writing the alternative form of the work-energy equation, viz. $\Delta(\mathrm{KE} + \mathrm{PE}) = \mathcal{W}_{1-2}$, and applying the data of the problem, we have

$$300\times 2\sin 20 - 1000\times 2 + \frac{1}{2}\left(\frac{1300}{9.81}\right)V^2 = -0.2\times 300\cos 20\times 2,$$

the solution of which provides the answer $V = \underline{5.0384}$ m/s.

Ex: 2.11 A body A is released from rest on a vertical circular path as shown in Fig. 2.13. If a constant resistance force of 1 N acts along the path, what is the speed of the body when it reaches B? The mass of A is 0.5kg and r = 1.6 m.

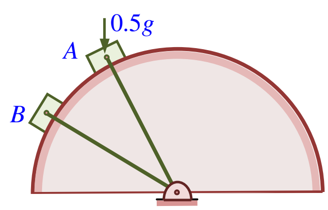

Figure 2.13

From the work-energy equation, we have

$$\left(0 - 0.5\times g\times 1.6\times(\sin 60 - \sin 30)\right) + \left(\tfrac{1}{2}0.5V^2 - 0\right) = 1\times 1.6\,\frac{\pi}{6},$$

from which we get the solution as $V = \underline{2.8523}$ m/s.

Ex: 2.12 A 10kN car starts from rest at A and moves without friction down the track shown in Fig. 2.14. (a) Determine the force exerted by the track on the car at point B, where the radius of the curvature is 7m; and (b) determine the minimum safe value of the radius of curvature at C.

First find V_B and V_C using the energy method. Then using the free body diagrams corresponding to positions B and C, write the equations of motion to obtain N_B and R_C (such that $N_C = 0$).

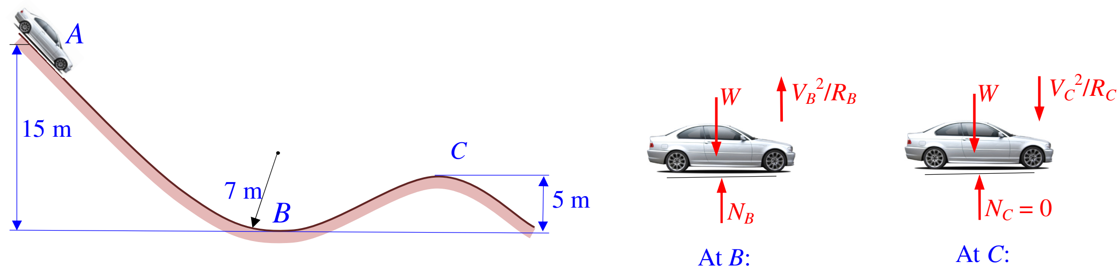

Figure 2.14

Ex: 2.13 A spring is used to stop a 60 kg package which is sliding on a horizontal surface as depicted in Fig. 2.15. The spring count k = 20 kN/m, and is held by cables so that it is initially compressed 120 mm. Knowing that the package has a velocity of 2.5 m/s in the position shown and that the maximum additional deflection of the spring is 40 mm, determine (a) μ_d between the package and surface, and (b) the velocity of the package as it passes again through the position shown.

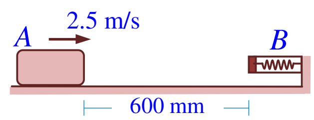

Figure 2.15

(a) From position A to B:

$$\left(0-\tfrac{1}{2}\times 60\times 2.5^2\right)+\left(\tfrac{1}{2}\times 20\times 10^3\times 0.160^2-\tfrac{1}{2}\times 20\times 10^3\times 0.120^2\right)=\mu_d\times 60\times 9.81\times 0.640\,,$$

from which we get μ_d = $\underline{0.2}$.

(b) From position B to A:

$$\left(\tfrac{1}{2}\times 60\times V^2-0\right)+\left(\tfrac{1}{2}\times 20\times 10^3\times 0.120^2-\tfrac{1}{2}\times 20\times 10^3\times 0.160^2\right)=0.2\times 60\times 9.81\times 0.640\,,$$

which yields the solution of V = $\underline{1.105}$ m/s.

System of Particles: Work energy Equation

Consider a system of n particles as shown in Fig. 2.16. We can write the work-energy equation for the i^{th} particle as

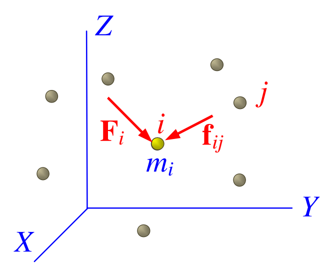

Figure 2.16

$$\int_1^2 \mathbf{F}_i \bullet d\mathbf{r}_i + \int_1^2 \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{f}_{ij} \right\} \bullet d\mathbf{r}_i = \left(\tfrac{1}{2} m_i V_i^2\right)_2 - \left(\tfrac{1}{2} m_i V_i^2\right)\,, \qquad [A]$$

where $\mathbf{F}_i$ is the external force acting on the i^{th} particle, and $\mathbf{f}_{ij}$ is the force exerted by the j^{th} particle on the i^{th} particle. The above equation can be written as

External Work +Internal Work = Change in Kinetic Energy relative to XYZ [B] for a displacement between $\mathbf{r}_1$ and $\mathbf{r}_2$ along some path. Moreover, we can identify conservative forces, both external internal, and utilise the potential energies for these forces. We can thus sum equation [A] for all particles; however, even though $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$, the work done due to these two equal and opposite forces may *not* cancel off as the movements may not be the same. On the other hand, in the case of rigid bodies, the internal work is zero. In the case of system of rigid bodies interconnected by pin or ball joints, if no friction exists; there will be no internal work. Thus, for a system of particles, w can write

$$\Delta(\mathrm{KE} + \mathrm{PE}) = \mathcal{W}_{1-2} \ ,$$

where $\mathcal{W}_{1-2}$ represents the work done by both the external and internal nonconservative forces and PE represents the total potential energy of conservative internal and external forces.

Important Note:

- Work done due to each force is due to the movement of the point of application of the force. That is, the forces move with their points of applications.
- Both internal and external forces may be conservative or nonconservative.
- Kinetic energy must be the total kinetic energy and not just that of the centre of mass.

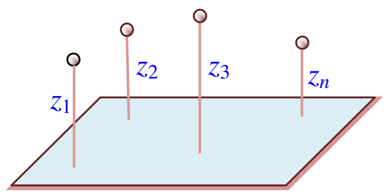

Figure 2.17

If there are several particles situated at different heights z_i as depicted in Fig. 2.17, we can write the potential energy of the system as

$$\mathrm{PE} = \sum_{i=1}^{n} m_i g\, z_i = W z_c \,,$$

where z_c is the height to the centre of gravity point (or centre of mass), if g is constant.

Kinetic Energy Expression Based on the Centre of Mass

Consider a system of n particles as shown in Fig. 2.18. The total kinetic energy of the system is given by

$$\mathrm{KE} = \sum_{i=1}^{n} \tfrac{1}{2} m_i V_i^2$$

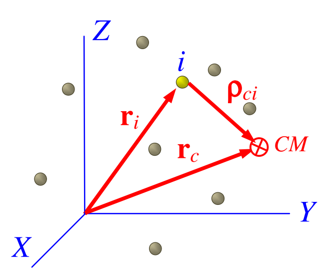

Figure 2.18

Denoting the position vector of the centre of mass by $\mathbf{r}_c$, we have

$$\mathbf{r}_i = \mathbf{r}_c + \boldsymbol{\rho}_{ci} \,.$$

Differentiating the above with respect to time, we get

$$\mathbf{V}_i = \mathbf{V}_c + \dot{\boldsymbol{\rho}}_{ci} ,$$

where $\mathbf{V}_i = \dot{\mathbf{r}}_i$, $\mathbf{V}_c = \dot{\mathbf{r}}_c$, and $\dot{\boldsymbol{\rho}}_{ci}$ is the velocity of the i^{th} particle relative to the centre of mass. Using these in the kinetic energy equation, we obtain

$$\text{KE} = \sum_{i=1}^{n} \tfrac{1}{2} m_i \left(\mathbf{V}_c + \dot{\boldsymbol{\rho}}_{ci}\right)\bullet\left(\mathbf{V}_c + \dot{\boldsymbol{\rho}}_{ci}\right) = \tfrac{1}{2}\sum_{i=1}^{n} m_i V_c^{\,2} + \sum_{i=1}^{n} m_i\, \mathbf{V}_c \bullet \dot{\boldsymbol{\rho}}_{ci} + \tfrac{1}{2}\sum_{i=1}^{n} m_i\, \dot{\rho}_{ci}^{\;2}$$

$$= \tfrac{1}{2} M\, V_c^{\,2} + \mathbf{V}_c \bullet \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \boldsymbol{\rho}_{ci}\right) + \tfrac{1}{2}\sum_{i=1}^{n} m_i\, \dot{\rho}_{ci}^{\;2} ,$$

which boils down to

$$\text{KE} = \tfrac{1}{2} M\, V_c^{\,2} + \tfrac{1}{2}\sum_{i=1}^{n} m_i\, \dot{\rho}_{ci}^{\;2} ,$$

as $\Sigma_i\, m_i\, \boldsymbol{\rho}_i$, is the first moment of mass of the system about the centre of mass which is zero. Thus, the kinetic energy for some reference can considered to be composed of (i) the kinetic energy of the total mass moving relative to that reference with the velocity of the centre of mass, and (ii) the kinetic energy of the motion of the particles relative to the centre of mass.

Ex: 2.14 A thin uniform hoop of radius R shown in Fig. 2.19 rolls without slipping such that O, the centre of mass, moves at a speed of V. If the hoop weighs W N, determine the kinetic energy of the hoop relative to ground.

The kinetic energy of the hoop can be written as

$$\text{KE} = \tfrac{1}{2}\frac{W}{g}V^2 + (\text{KE})_{\text{relative to centre of mass}} .$$

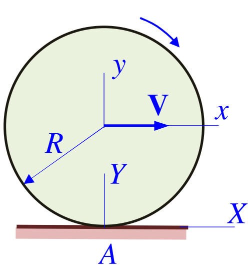

Figure 2.19

The point of contact A has zero instantaneous velocity with respect to ground (due to no slipping condition). The hoop has a pure instantaneous rotational motion about A. The angular velocity being given by $\omega = V/R$. As xy translates with respect to XY, an observer on xy sees the same angular velocity ω for the hoop as the observer on XY. Therefore, the second term in the above kinetic energy equation can be obtained as:

$$(\text{KE})_{\text{relative to centre of mass}} = \tfrac{1}{2}\int_0^{2\pi}\left(\frac{W}{2\pi Rg}\right) R\,d\theta = \tfrac{1}{2}\frac{W}{2\pi\, g}V^2 2\pi = \tfrac{1}{2}\frac{W}{g}V^2 .$$

Hence, the total kinetic energy of the hoop can be written as

$$\text{KE} = \tfrac{1}{2}\frac{W}{g}V^2 + \tfrac{1}{2}\frac{W}{g}V^2 = \frac{W}{g}V^2 .$$

Ex: 2.15 Determine the total kinetic energy of the blocks shown in Fig. 2.20 when block B falls down by x_B. The cord is flexible with a spring constant of K_2.
Writing the work-energy equation

$$\Delta \mathrm{PE} + \Delta \mathrm{KE} = \mathcal{W}_{1-2}\,.$$

In the above,

$$\Delta \mathrm{PE} = \left(\tfrac{1}{2}K_1 x_A{}^2 + \tfrac{1}{2}K_2\left(x_B - x_A\right)^2 - 0\right) + \left(-W_B \times x_B - 0\right)\,,$$

$$\Delta \mathrm{KE} = \tfrac{1}{2}m_A V_A{}^2 + \tfrac{1}{2}m_A V_A{}^2 - 0\,,$$

and

$$\mathcal{W}_{1-2} = -W_A \times \mu_d \times x_A\,.$$

It is possible to calculate ΔKE knowing various data in the problem from the first equation.

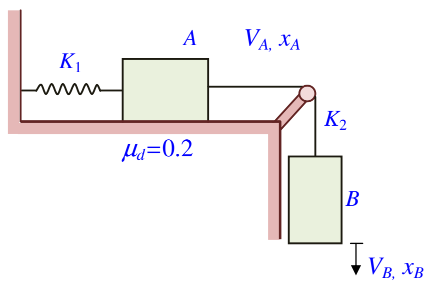

Figure 2.20

Ex: 2.16 A hypothetical vehicle shown in Fig. 2.21 is moving at speed V_0. There are two bodies of mass m each sliding along a horizontal rod at a speed of v relative to the rod. The rod is rotating at an angular speed of ω rad/s relative to the vehicle. Find the kinetic energy of the two bodies relative to the ground XYZ when they are at a distance r apart.

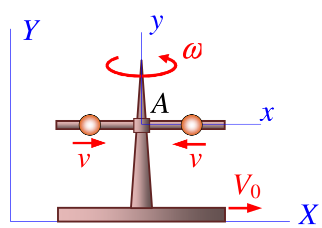

Figure 2.21

The centre of mass is A, and the kinetic energy about it is given by

$$\tfrac{1}{2}M\,V_c{}^2 = \tfrac{1}{2}(2\mathrm{m})V_c{}^2\,.$$

The velocity of each ball relative to xyz, (which translates with respect to XYZ) can be obtained using cylindrical coordinates as $\mathbf{r} = r\,\varepsilon_r$ and $\mathbf{V} = \dot{r}\,\varepsilon_r + r\dot{\theta}\,\varepsilon_\theta$. Hence,

$$\dot{\rho}^2 = \dot{r}^2 + (r\omega)^2 = v^2 + (r\omega)^2\,.$$

Therefore, we get the total kinetic energy as

$$\mathrm{KE} = mV_0{}^2 + m\left(v^2 + r^2\omega^2\right).$$

Work-Kinetic Energy Expression Based on Centre of Mass

We have seen earlier that the Newton's law for the centre of mass for a system of particles is given by:

$$\mathbf{F} = M\,\ddot{\mathbf{r}}_c,$$

where **F** is the external force acting on the system of particles. As before, we can deduce from the above that

$$\int_1^2 \mathbf{F}\bullet d\mathbf{r}_c = \left(\tfrac{1}{2}M\,V_c^2\right)_2 - \left(\tfrac{1}{2}M\,V_c^2\right)_1 .$$

The external forces must all move with the centre of mass for computing the work done term. Single particle model is a special case of the above equation. That is, the single particle model represents the case where the motion of the centre of mass of a body sufficiently describes the motion of the body and where the external forces on the body essentially move with the centre of mass of the body.

Key features of this approach:

- Only external forces are involved.
- All forces move with the centre of mass while computing work done.
- The kinetic energy of the centre of mass only is involved.

Ex: 2.17 A cylinder of total mass M and radius R rotates about its axis with an angular speed of W. Determine its kinetic energy.

The kinetic energy is obtained by considering an elemental mass dm as depicted in Fig. 2.22 to obtain

$$\text{KE} = \frac{1}{2}\int_m dm\,(r\omega)^2 = \omega^2 \int_0^{2\pi}\int_0^R r\,dr\,d\theta\,\rho l\,r^2 = \frac{R^4\omega^2}{4}2\pi\rho l = \frac{MR^2\omega^2}{2}.$$

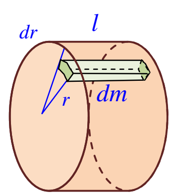

Figure 2.22

Ex: 2.18 A cylinder of mass 25 kg and diameter 0.6 m rolls without slipping as shown in Fig. 2.23. Find the speed after it rolls 1.6 m down the incline. Also determine the frictional force acting on the cylinder.

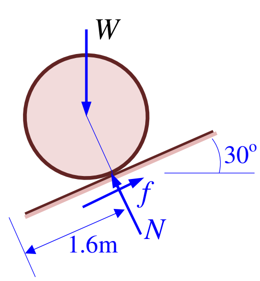

Figure 2.23

A. SYSTEM OF PARTICLES APPROACH:

We have the work-energy equation applied to the cylinder considering it as a system of particles to obtain

$$\Delta(\mathrm{PE}+\mathrm{KE}) = \mathcal{W}_{1-2},$$

which leads to

$$(0-25\mathrm{g}\times 1.6\sin 30)+\left(\frac{1}{2}25\times V_c^{\,2}+\frac{1}{4}\times 25\times 0.3^2\times\omega^2-0\right)=0\,.$$

Note that only particles on the rim are acted upon by the external forces N and f. Each particle on the rim has N and f only when the particle is in contact with the surface; then the velocity of that point is zero. Hence the frictional force f does no work in this approach as it acts on one particle at an instant of time and in the next instant it is zero on that particle. The velocity of the centre of cylinder, V_c, is related to the angular velocity ω by

$$V_c = R\omega.$$

Hence, we obtain from the above that V_c = 3.23 m/s.

B. CENTRE OF MASS APPROACH:

Now, in order to find f, the frictional force, let us employ the centre of mass approach. Now all the external forces must move with the centre of mass. Thus f does work. Writing the work-energy equation, now considering the kinetic energy of the centre of mass alone, we obtain

$$-f\times 1.6+25g\times 1.6\sin 30=\tfrac{1}{2}MV_c^{\,2},$$

which on substitution yields the answer f = 41.1 N.

Ex: 2.19 An external torque T of 50 Nm is applied to a solid cylinder B as shown in Fig. 2.24 which has a mass of 30 kg and radius of 0.2 m. The cylinder rolls without slipping. Block A having a mass of 20 kg is dragged up the 15^0 inline. The coefficient of dynamic friction $\mu_d = 0.25$ between A and the incline. Connections at C and D are frictionless. (a) What is the velocity of the system after moving a distance, d = 2m? (b) What is the frictional force f on the cylinder?

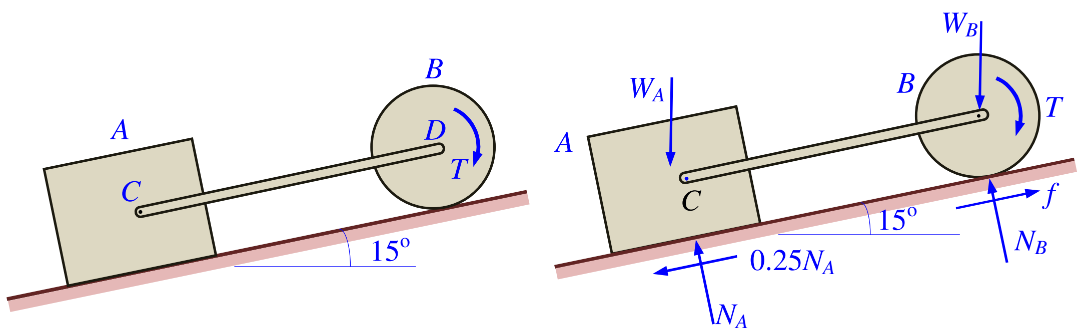

Figure 2.24

A. SYSTEM OF PARTICLES APPROACH:

The pairs of forces in CD are equal and opposite, and move by the same distance; hence zero internal work. Writing the work-energy equation, we get

$$\mathcal{W}_{1-2} = \Delta\mathrm{PE}+\Delta\mathrm{KE}\,.$$

The above on substitution leads to

$$T\theta-(\mu_d N_A)d=[(W_A+W_B)d\sin 15-0]+\left[(m_A+m_B)\frac{V^2}{2}+\frac{1}{4}m_B r_B^{\,2}\,\omega_B^{\,2}-0\right],$$

where the distance d moved by the cylinder during an angular movement θ is $R\theta = d$. In addition, $V = R\omega$. Substitution of these in the above equation yields V = 2.158 m/s.

B. CENTRE OF MASS APPROACH:

Using the centre of mass approach, the torque T during the translation does no work. Hence, we obtain

$$\int_1^2 \mathbf{F} \bullet d\mathbf{r}_c = \frac{1}{2}(m_A + m_B)\left(V_2^2 - V_1^2\right).$$

That is

$$-0.25 N_A d + fd - (W_A + W_B)\sin 15 d = \frac{1}{2}(m_A + m_B)\left(V^2 - 0\right),$$

which yields the frictional force f = <u>232.5</u> N.

Ex: 2.20 Find the angular speed of the cylinder shown in Fig. 2.25 after it rotates 20° starting from rest. The spring is originally unstretched. The cylinder rolls down without slipping. K = 500 N/m. m_A = 30 kg and d_A = 0.4 m.

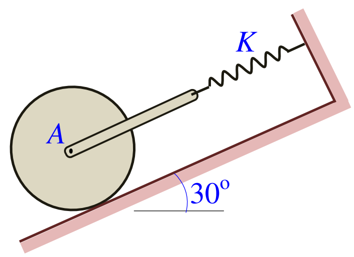

Figure 2.25

SYSTEM OF PARTICLES APPROACH:

The work-energy equation can be written as

$$\Delta(\text{PE} + \text{KE}) = \mathcal{W}_{1-2},$$

which yields

$$\tfrac{1}{2} mV^2 + \tfrac{1}{4} mR^2\omega^2 + \tfrac{1}{2} Kx^2 - 30g\ x\sin 30 = 0.$$

Since $V = R\omega$, $x = R\theta$ = 0.2×20×π/180, we get from the above the solution as ω = <u>3.1718</u> rad/s.

Methods of Momentum for Particles

LINEAR MOMENTUM OF A PARTICLE

Newton's Law can be mathematically stated as

$$\mathbf{F} = \frac{d(m\mathbf{V})}{dt} = \frac{d\mathbf{P}}{dt},$$

where $\mathbf{P} = m\mathbf{V}$ is called the *linear momentum vector* of the particle.

IMPULSE AND MOMENTUM RELATIONS FOR A PARTICLE

Multiplying both sides of the Newton's law statement given above by dt and integrating from and initial time t_i to a subsequent time t_f, we obtain

$$\int_{t_i}^{t_f} \mathbf{F}\, dt = \int_{t_i}^{t_f} m\frac{d\mathbf{V}}{dt}\, dt,$$

or

$$\int_{t_i}^{t_f} \mathbf{F}\, dt = m\mathbf{V}_f - m\mathbf{V}_i.$$

In the above, the term on the left hand side

$$\mathbf{I} = \int_{t_i}^{t_f} \mathbf{F}\, dt$$

is called the *impulse* of the force **F** during the time interval $t_i - t_f$. The right hand side represents the change in the linear momentum vector during the time interval. That is the impulse over a time interval equals the change in linear momentum of the particle during that time interval. The following points are worth pondering upon:

- The impulse of the force may be known even though the force itself may not be known.
- To produce an impulse, a force needs to exit for an interval of time. Its point of application may not move.

Ex: 2.21 A particle initially at rest is acted upon by a force whose variation with time is as shown in Fig. 2.26. If the particle has a mass of 2 kg and is constrained to move rectilinearly along the force, what is its velocity at 2 sec?

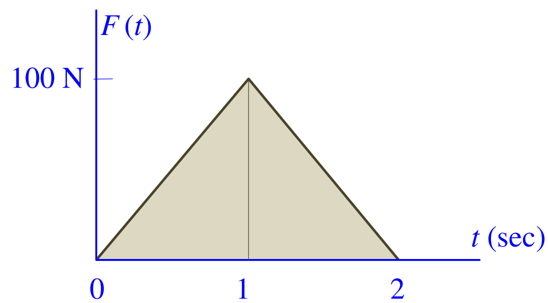

Figure 2.26

The impulse of the force is $\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt = 100$ Ns $= mV_2 - mV_1 = 2\times V_2$. Hence, $V_2 = 100/2 = \underline{50}$ m/s.

The impulse momentum principle is exceptionally useful when $\mathbf{F}(t)$ is not known mathematically and instead is given graphically. Then we can determine the area of the graph (using, for example, a planimeter), thus getting an easy and quick solution.

Ex: 2.22 A particle of mass 1 kg is initially at rest at the origin of a reference. A force $\mathbf{F}(t) = t^2\,\mathbf{i} + (6t + 10)\,\mathbf{j} + 1.6t^3\,\mathbf{k}$ kN acts on the particle, where t is in seconds. Find the velocity after 10 seconds.

Employing the impulse momentum principle, we obtain

$$m\mathbf{V}_2 - m\,\mathbf{0} = \int_0^{10} \mathbf{F}\, dt$$

from which we get the velocity after 10s as, $\mathbf{V}_2 = \frac{10^3}{3}\mathbf{i} + \left(\frac{600}{2} + 100\right)\mathbf{j} + 1.6\times\frac{10^4}{4}\mathbf{k}$ m/s.

Linear Momentum Considerations for a System of Particles

For a system of n particles, we can write

$$\mathbf{F} = \sum_{j=1}^{n} m_j \frac{d\mathbf{V}_j}{dt}.$$

Multiplying both sides by dt and integrating (as before), we obtain

$$\int_{t_i}^{t_f} \mathbf{F}\, dt = \left(\sum_j m_i \mathbf{V}_i\right)_f - \left(\sum_j m_i \mathbf{V}_i\right)_i \qquad \text{[A]}$$

In the above equation,

- **F** is the total external force (as the internal forces cancel off).
- Thus, the impulse of the total external forces on the system of particles during a time interval equals the sum of the changes of the linear momentum vectors of the particles during the time interval.

Ex: 2.23 A truck weighing 40 kN is moving at 70 km/hr as depicted in Fig. 2.27, when the driver suddenly applied brakes at time t = 0. Load A weighing 12 kN broke loose from its ropes and at t =3 s, was seen sliding at a speed of 1 m/s relative to the truck. Find the speed of the truck at this instance if the coefficient of dynamic friction between the tyres and the road is 0.4.

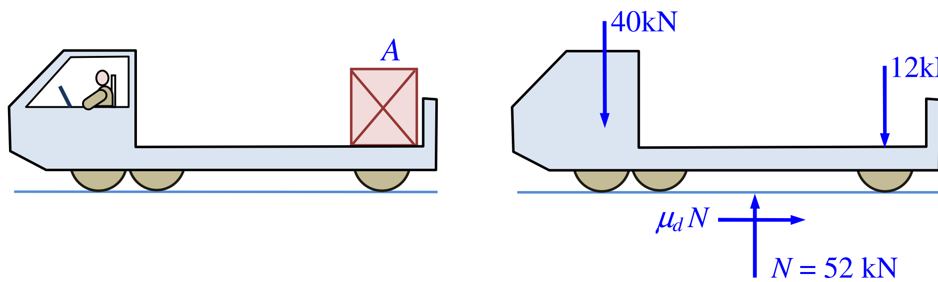

Figure 2.27

The various forces shown in the free body diagram are N = 40 + 12 = 52 kN and $\mu_d N$ = 20.8 kN. Applying the impulse-momentum principle, we obtain

$$\int_0^3 (-20.8)\times 10^3\, dt = \frac{40\times10^3}{g}V + \frac{12\times10^3}{g}(V+1) - \frac{52\times10^3}{g}\times\frac{70\times10^3}{3600},$$

from which we obtain the solution as V = 7.44 m/s = 26.8 km/hr.

Introducing the centre of mass into Eq. [A] is advantageous at times. Thus, we have

$$M\mathbf{r}_c = \sum_i m_i \mathbf{r}_i ,$$

which on differentiation with respect to time yields

$$M\,\mathbf{V}_c = \sum_i m_i \mathbf{V}_i .$$

Thus, the total linear momentum of a system of particles equals the linear momentum of a particle that has the total mass of the system and that moves with the velocity of the mass centre. Applying this principle in Eq. [A] provides

$$\int_{t_i}^{t_f} \mathbf{F}\, dt = M(\mathbf{V}_c)_f - M(\mathbf{V}_c)_i . \qquad \text{[B]}$$

Single particle model can be thus seen as a special case of the system of particles. Depending on whether it is easy to locate the centre of mass or not, one may employ [A] or [B].

Ex: 2.24 Two adjacent tanks A and B are shown in Fig. 2.28. Both are rectangular with a width of 4 m. Petrol from A is being pumped into B. When the level of tank is 0.7 m from top, the rate of flow from A to B is 300 litres/s. Ten seconds later it is 500 litres/s. What is the horizontal force from the fluid on to the tank during this 10 seconds time interval? Density of petrol is 0.8×10^3 kg/m^3. Tank A is originally full and tank B is originally empty.

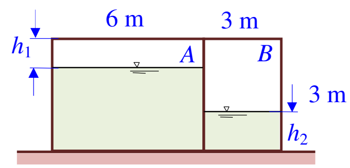

Figure 2.28

At time t, let h_1 be the drop of petrol in tank A and h_2 be the corresponding rise in tank B. Therefore, we have:

$$h_1\times6\times4 = h_2\times3\times4 \text{ or } h_2 = 2\,h_1.$$

Location of the centre of mass: Taking moments of mass of petrol about the left edge of tank A, we get:

$$M\,x_c = M_A\,x_A + M_B\,x_B,$$

where x_c is the distance of centre of mass from the left edge of tank A. That is:

$$6\times3\times4\times\rho\times x_c = (3-h_1)\times6\times4\times3\times\rho + h_2\times3\times4\times\rho\times7.5.$$

Differentiating the above with respect to t, we obtain

$$\dot{x}_c = \frac{1}{6\times3\times4}\{-3Q + Q\times7.5\},$$

where Q is the rate of flow from tank A to B. That is $Q = 6\times4\times\dot{h}_1$. From the above we get $\dot{x}_c = \dfrac{Q}{16}$.

Now, writing the impulse momentum equation, we have

$$\int_0^{10}\mathbf{F}\,dt = (M\dot{x}_c)_2 - (M\dot{x}_c)_1,$$

which on substitution yields

$$F_{av}\times10 = 6\times3\times4\times\rho\times\{(Q_2-Q_1)/16\},$$

from which we get the solution as $F_{av} = \underline{72}$ N.

Conservation of Linear Momentum

If the total external force on a system of particles is zero, the change in linear momentum is zero. Moreover, if the external force continues to be zero (i.e. the impulse is zero), the centre of mass remains stationary, if the velocity of the centre of mass is zero at some instance of time.

Impulsive Forces

Forces that act over a very short time such as the one shown in Fig. 2.29, but have appreciable impulse are called *impulsive forces*. One good example is the explosive loads.

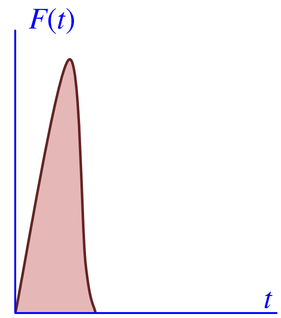

Figure 2.29

Due to impulsive force a particle may change its velocity while there is hardly any change in its position[1]. Consider a bomb suspended from the ceiling as depicted in Fig. 2.30. After the explosion there is hardly any change in the position of the centre of mass as there is no external force acting—although the flying fragments may have high velocities (gravity may be ignored during the short interval of time Δt).

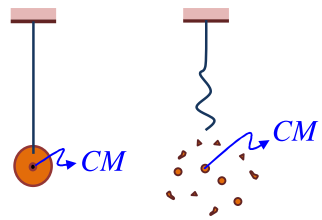

Figure 2.30

Ex: 2.25 A tennis player during his serve is able to hit the ball when the ball is tossed up and reaches a stationary position. He hits the ball with such a force that it starts flying as shown in Fig. 2.31 with a speed of 150 km/hr. Estimate the average force imparted by his racquet on the ball if the contact time during the serve was 0.005 seconds. The mass of the ball is 58 grams.

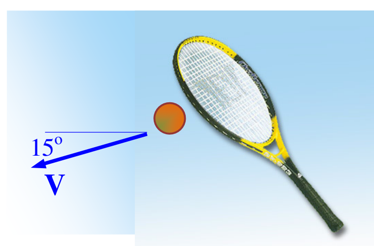

Figure 2.31

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[1] Since

$$\int_0^{\Delta t} F\,dt \approx F_{ar}\,\Delta t = mV_{\max}\ ,$$

we have

$$V_{\max} = \frac{F_{Ar}}{m}\Delta t\,.$$

On the other hand,

$$x_{\max} = \int_0^{\Delta t} V_{\max}\,dt = \frac{F_{Ar}}{m}\Delta t\ \Delta t\,,$$

which is a higher order term when compared to V_{max}.

We have $\mathbf{F}_{av}\ \Delta t = m\ \mathbf{V}$. Therefore $\mathbf{F}_{av} = 0.058\times(150\times10^3/3600)(\cos 15\ \mathbf{i} - \sin 15\ \mathbf{j})/0.005 = (466.864\ \mathbf{i} - 125.096\ \mathbf{j})$ N. Magnitude of this average force is F_{av} = <u>483.333</u> N.

Ex: 2.26 A cannon of weight 9 kN fires a 45 N projectile with a muzzle velocity of 625 m/s at an angle of 50° as depicted in Fig. 2.32. Find the maximum compression in the spring if K = 4 kN/m.

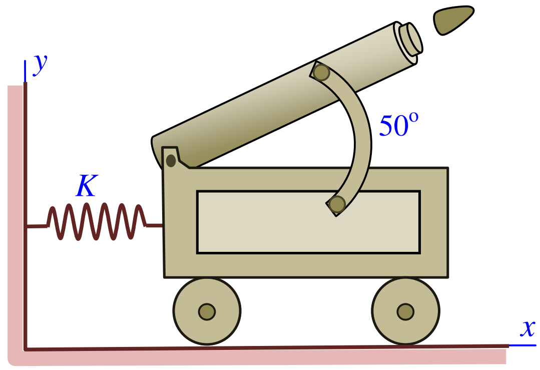

Figure 2.32

The firing takes place in a very short time interval; the force on the projectile and the force on the cannon are both impulsive. Thus, the cannon achieves an instantaneous recoil velocity without much of a displacement. The linear momentum is conserved as there is no external force. Thus, we can write the momentum equation along the x-direction as

$$(mV_x)_{\text{cannon}} + (mV_x)_{\text{projectile}} = 0\,.$$

If V_c is the velocity of the cannon along the x-direction and V_p that of the projectile, we have

$$V_p = V_c + 625\cos 50\,.$$

Substituting in the momentum equation, we get

$$\frac{9000}{g}V_c + \frac{45}{g}\left(V_c + 625\cos 50\right) = 0\,,$$

from which we obtain $V_c = -2$ m/s.
After this impulsive action which results in the above instantaneous velocity of cannon, we can use conservation of mechanical energy to solve the question. Thus we can write Δ(KE + PE) = 0, which leads to

$$\left(0 - \frac{1}{2}\frac{9000}{g}2^2\right) + \left(\frac{1}{2}\times 4000\delta^2 - 0\right) = 0\,,$$

which yields the solution δ = <u>0.958</u> m.

Ex: 2.27 A 1300 kg jeep with 3 persons of mass 100 kg each is being tested to see what is the maximum velocity which it can reach on an icy road in 5 seconds. If $\mu_s = 0.1$, find V_{max}.

The impulse momentum equation yields

$$\int \mathbf{F}\,dt = mV - 0\,.$$

That is $1600g \times 0.1 \times 5 = 1600V_{\max}$. Hence, V_{max} = <u>4.905</u> m/s (= <u>17.66</u> km/hr).

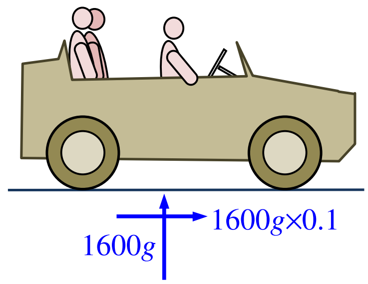

Figure 2.33

Impact

Consider two bodies which collide but do not break. The time interval of impact is usually very small. However, large impulsive forces are set in which are equal and opposite. The total linear momentum before impact equals the total linear momentum after impact.

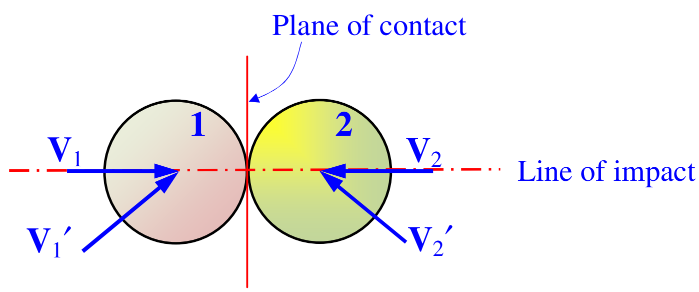

Figure 2.34

Central impact: if the centres of the two masses lie along the line of impact (which is normal to the plane of contact), the impact is said to be central. Otherwise it is *eccentric*.
Direct central impact: is a central impact in which the velocity vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are collinear with the line of impact.
Oblique central impact: is one in which one or both the velocity vectors are not collinear with the line of impact.
In any of the above cases, the linear momentum is conserved during the short time of impact. Thus, we have

$$(m_1\mathbf{V}_1)_i + (m_2\mathbf{V}_2)_i = (m_1\mathbf{V}_1)_f + (m_2\mathbf{V}_2)_f .$$

In the case of direct central impact, we need one further equation (we have just the above equation, which turns out to be a scalar equation for this case). In the case of oblique impact, if we know the initial velocity components, we have six unknown velocity components, but only three equations.

CASE: 1. DIRECT CENTRAL IMPACT:

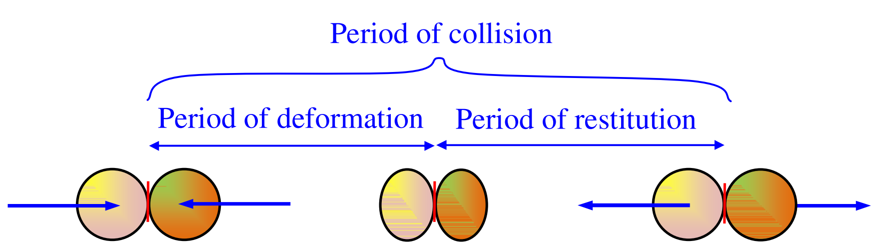

Figure 2.35

Consider Fig. 2.35. The total period of collision is composed of (i) period of deformation in which the two bodies start deforming from an initial undeformed state, and (ii) a period of restitution in which the bodies start recovering from their deformed state. In the case of perfectly elastic impact the recovery is complete as depicted in Fig. 2.35; whereas in the case of inelastic impact the recovery is partial as shown in Fig. 2.36. In a perfectly plastic impact, the two bodies stick together and move together after impact.

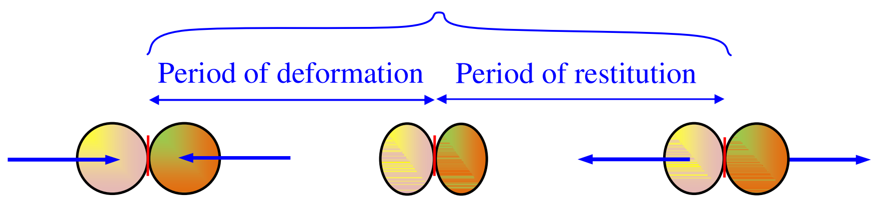

Figure 2.36

The impulse during the period of deformation and restitution could be represented by $\int D\,dt$ and $\int R\,dt$ respectively. The coefficient of restitution is then defined as the ratio of these impulses. Thus, we have the coefficient of restitution

$$\varepsilon = \frac{\int R\,dt}{\int D\,dt}.$$

The coefficient of restitution ε depends on the material and also on the size, shape and approach velocities of the two bodies. However, ε for different materials have been established (without considering other factors) and can be used for getting approximate results.
For body 1, if $(\mathbf{V}_1)_D$ is the velocity at maximum deformation, we have

$$\int D\,dt = (m_1V_1)_D - (m_1V_1)_i = -m_1\left[(V_1)_i - (V_1)_D\right].$$

During the period of restitution, we can likewise write

$$\int R\,dt = -m_1\left[(V_1)_D - (V_1)_f\right],$$

from which we obtain

$$\varepsilon = \frac{(V_1)_D - (V_1)_f}{(V_1)_i - (V_1)_D}. \qquad \text{[A]}$$

Similarly, for body 2, we have

$$\varepsilon = \frac{(V_2)_0 - (V_2)_f}{(V_2)_i - (V_2)_D} = \frac{(V_2)_f - (V_2)_D}{(V_2)_D - (V_2)_i}. \qquad \text{[B]}$$

At the intermediate position—that is at the end of the deformation period and just at the start of restitution, the two bodies have the same velocity. Thus, we have

$$(V_1)_D = (V_2)_D.$$

Now, since if $\frac{a}{b} = \frac{c}{d} = r$, we have $a = br$ and $c = dr$. Hence, $r = (a - c)/(b - d)$. From Eqs. [A] and [B], we obtain the coefficient of restitution as

$$\varepsilon = -\frac{(V_2)_f - (V_1)_f}{(V_2)_i - (V_1)_i} = -\frac{\text{relative velocity of seperation}}{\text{relative velocity of approach}}.$$

This equation along with the linear momentum equation can be used to solve problems.

For a perfectly elastic collision, the impulse for during the period of restitution equals the impulse during the period of deformation. Thus, restitution here is the reverse of deformation. Hence, $\varepsilon = 1$. In the case of an inelastic impact $\varepsilon < 1$ as the impulse is diminished during restitution as the bodies are unable to resume their original geometries. For a perfectly plastic impact $\varepsilon = 0$, and the two bodies continue to remain in contact as if they are glued.

CASE: 2. OBLIQUE CENTRAL IMPACT

Velocity components along the line of impact are related by the scalar components of the equation:

$$(m_1\mathbf{V}_1)_1 + (m_2\mathbf{V}_2)_i = (m_1\mathbf{V}_1)_f + (m_2\mathbf{V}_2)_f .$$

- Also e.g. [C] along the line of impact can be used ($\varepsilon -$ can be considered to be the same as that of direct central impact for smooth bodies).
- Thus, we can solve for the velocity components along the line of impact.
- For smooth bodies, the other rectangular components of the velocity are unaffected by the impact as no impulses act in their directions or either body.

Ex: 2.28 Two identical billiard balls collide with velocities as shown in Fig. 2.37. Find their final velocities and the loss in kinetic energy.

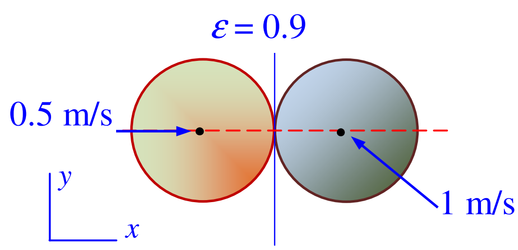

Figure 2.37

From Fig. 2.37, we have: $V_{1x}|_i = 0.5$ m/s, $V_{2x}|_i = -0.707$ m/s, and $V_{2y}|_i = 0.707$ m/s. Along the line of impact, the linear momentum before impact $= 0.5m - 0.707m = mV_{1x}\big|_f + mV_{2x}\big|_f$, where m is the mass of each ball.

The coefficient of restitution: $\varepsilon = 0.9 = -\dfrac{V_{2x}\big|_f - V_{1x}\big|_f}{-7.07 - 5}$.

Solving the above two equations simultaneously, we obtain $V_{1x}|_f = -0.647$ m/s and $V_{2x}|_f = 0.44$ m/s. Moreover, $V_{2y}|_f = V_{2y}|_i = 0.707$ m/s. Hence, the final velocities after impact are

$$\mathbf{V}_1|_f = \underline{-0.647\,\mathbf{i}} \text{ m/s and } \mathbf{V}_2|_f = \underline{0.44\,\mathbf{i} + 0.707\,\mathbf{j}} \text{ m/s.}$$

The loss in kinetic energy is obtained as

$$KE\big|_i - KE\big|_f = \tfrac{1}{2}m0.5^2 + \tfrac{1}{2}m(1)^2 - \tfrac{1}{2}m\,0.647^2 - \tfrac{1}{2}m\left(0.44^2 + 0.707^2\right) = \underline{0.0689\,m} \text{ Nm.}$$

Ex: 2.29 A 50 kg block is dropped onto a 20 kg pan of a spring scale from a height of 500 mm as shown in Fig. 2.38. Assuming the impact to be perfectly plastic, determine the maximum deflation of the ran $K = 30$ kN/m.

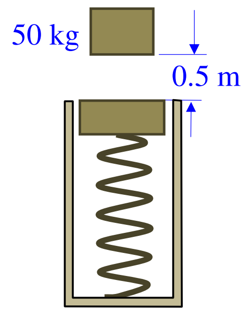

Figure 2.38

Velocity at which the block hits the pan can be obtained by using the work-energy equation as:

$$\left(\tfrac{1}{2}mV^2 - 0\right) + \left(0 - 0.5 \times 50 \times g\right) = 0,$$

from which we get V = 3.132 m/s.

Now, the velocity with which the pan and block start moving downward can be obtained as (note that it is a perfectly plastic impact) $50 \times 3.132 + 20 \times 0 = 70\ V_c$. Therefore, V_c = 2.237 m/s.

Lastly, in order to find the maximum deflection of the pan, we use the work-energy equation again to get

$$\left(0 - \tfrac{1}{2} \times 70 \times 2.237^2 - \tfrac{1}{2} 70 g \times \delta\right) + \left(\tfrac{1}{2} \times 30 \times 10^3 \times \delta^2\right) = 0.$$

Solving the above quadratic equation in δ we get the answer as δ = 0.120 mm.

Moment of Momentum Equation for a Single Particle

The Newton's Law can be written as

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{V}) = \dot{\mathbf{P}},$$

where $\mathbf{P} = m\mathbf{V}$ is the linear momentum of the particle. Consider Fig. 2.39 in which a particle is seen moving along a trajectory. Taking moments of the above equation about the point A, we obtain

$$\boldsymbol{\rho}_a \times \mathbf{F} = \boldsymbol{\rho}_a \times \dot{\mathbf{P}}.$$

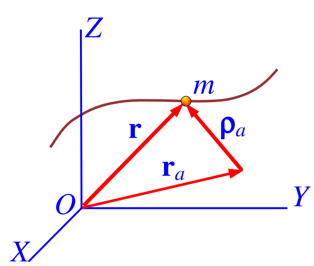

Figure 2.39

Consider

$$\frac{d}{dt}(\boldsymbol{\rho}_a \times \mathbf{P}) = \boldsymbol{\rho}_a \times \dot{\mathbf{P}} + \dot{\boldsymbol{\rho}}_a \times \mathbf{P}.$$

Now, $\mathbf{r} = \boldsymbol{\rho}_a + \mathbf{r}_a$. Let A be fixed in location. Hence we have $\dot{\mathbf{r}} = \dot{\boldsymbol{\rho}}_a$. Therefore, the second term on the right hand side of the above equation reduces to naught as

$$\dot{\boldsymbol{\rho}}_a \times \mathbf{P} = \dot{\boldsymbol{\rho}}_a \times m\dot{\mathbf{r}} = \dot{\boldsymbol{\rho}}_a \times m\dot{\boldsymbol{\rho}}_a = \mathbf{0}.$$

Thus we have

$$\frac{d}{dt}(\boldsymbol{\rho}_a \times P) = \boldsymbol{\rho}_a \times \dot{\mathbf{P}}.$$

Using the above in the earlier equation we get

$$\boldsymbol{\rho}_a \times \mathbf{F} = \frac{d}{dt}(\boldsymbol{\rho}_a \times \mathbf{P}),$$

which can otherwise be written as

$$\mathbf{M}_a = \dot{\mathbf{H}}_a.$$

In the above, $\mathbf{H}_a$ is known as the *moment of momentum vector* about a, or the *angular momentum vector*. The above equation thus corresponds to: "the moment of the external force acting on the particle about a fixed point a in an inertial frame of reference $\mathbf{M}_a$ equals the time rate of change of moment of momentum of the particle about a".

A component form of the above equation could be written, for example, as

$$M_z = \dot{H}_z,$$

where M_z is the moment of external force about the z-axis, and H_z is the moment of momentum again about the z-axis.

If $\mathbf{M}_a = \mathbf{0}$, it follows that $\mathbf{H}_a$ remains a constant. This is the *law of conservation of moment of momentum principle*.

Ex: 2.30 A small ball weighing 1 kg is rotating about a vertical axis at a speed of ω_1 =10 rad/s as shown in Fig. 2.40. The ball is connected to bearings on the shaft by light inextensible strings having a length of 1 m each. The angle θ_1 = 30°. What is the value of angular velocity ω_2 of the ball if the bearing A is moved up by 200 mm?

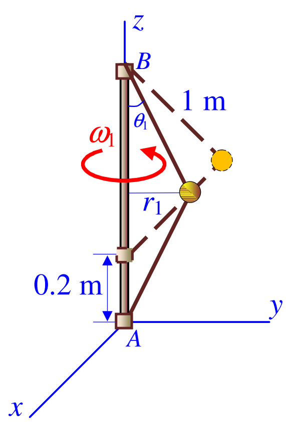

Figure 2.40

As, $M_z = 0$, H_z is a constant. Therefore $(H_z)_1 = (H_z)_2$, where

$$(H_Z)_1 = m(r_1\omega_1)r_1 \text{ and } (H_Z)_2 = m(r_2\omega_2)r_2.$$

Since θ_1 = 30° and cos θ_2 = cos θ_1 − 0.1, w get θ_2 = 40°. Moreover, r_1 = 0.5 m and r_2 = 0.6428 m. Therefore we get $\omega_2 = r_1^2\,\omega_1/\,r_2^2$ = <u>6.05</u> rad/s.

Ex: 2.31 A small ball attached to the end of a string is supported by a smooth horizontal plane and travels with uniform speed V_0 in a circular path of radius r as shown in Fig. 2.41. By pulling the string at the lower end, the radius of the path is reduced to $r/2$. What is the new velocity of the ball? What is the tension T?

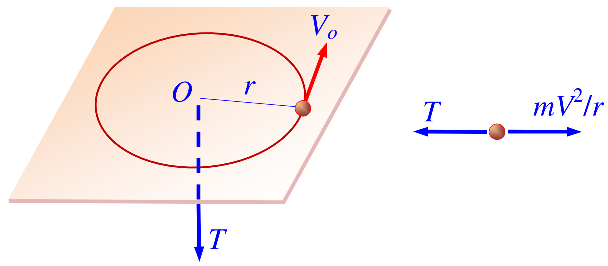

Figure 2.41

$(\mathbf{H}_0)_i = r\,mV_0$ and $(\mathbf{H}_0)_f = \frac{r}{2}mV_1$. Since $\mathbf{M}_0 = \mathbf{0}$, $\mathbf{H}_0$ must remain a constant. Thus, equating these two, we obtain $V_1 = 2\,V_0$. The tension $T = m(2V_0)^2/(r/2) = 8\,mV_0^2/r$.

Ex: 2.32 A conical pendulum of length $l = 2$ m rotates with constant angular speed ω in a horizontal circular path of radius $r = 100$ mm as shown in Fig. 2.42. How much string must be pulled through the pedestal to double the speed of the ball?

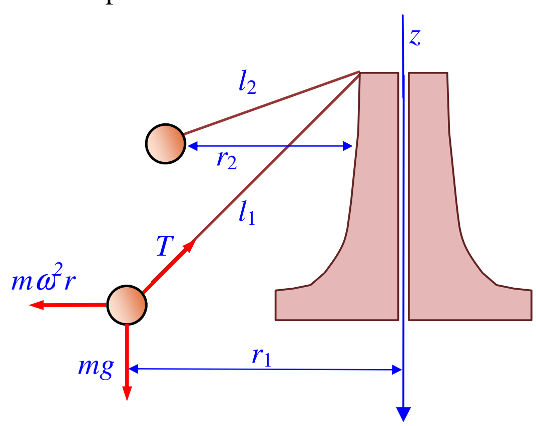

Figure 2.42

As $M_z = 0 = \dot{H}_z$, H_z must remain a constant. That is $(H_z)_i = (H_z)_f$. Now,

$$(H_z)_i = mr_1^2\omega_1 \text{ and } (H_z)_f = mr_2^2\omega_2 .$$

Therefore, $r_1^2\omega_1 = r_2^2(2\omega_1)$ from which we obtain $r_2^2 = \dfrac{100^2}{2}$ and $\omega_2 = 2\omega_1 = 0.475\,\text{rad/s}$.

Since $\tan\theta_2 = \dfrac{\omega_2^2 r_2}{g} = 1.633$, we get $\theta_2 = 58.52^\circ$. Therefore $l_2 = 82.91$ mm and the answer is $l_1 - l_2$ = 117.087 mm.

Moment of Momentum for a System of Particles

ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

Consider a system of n particles as shown in Fig. 2.43. Let c indicate the centre of mass of the particles at this instant of time and let a be a fixed point. We would like to derive an expression for the moment of momentum of the system of particles about a, viz. $\mathbf{H}_a$.

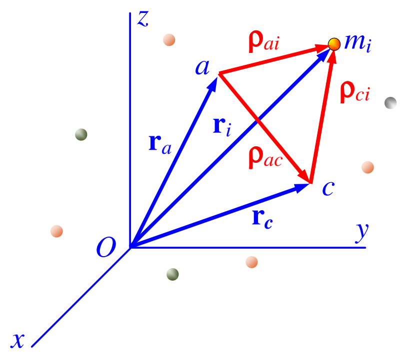

Figure 2.43

Thus we can write

$$\mathbf{H}_a = \sum_{i=1}^{n} \boldsymbol{\rho}_{ai} \times m_i \dot{\mathbf{r}}_i = \sum_i (\boldsymbol{\rho}_{ac} + \boldsymbol{\rho}_{ci}) \times m_i (\dot{\mathbf{r}}_c + \dot{\boldsymbol{\rho}}_{ci}), \qquad \text{[A]}$$

which follows from $\boldsymbol{\rho}_{ai} = \boldsymbol{\rho}_{ac} + \boldsymbol{\rho}_{ci}$ and $\mathbf{r}_i = \mathbf{r}_c + \boldsymbol{\rho}_{ai}$ as depicted in Fig. 2.43. Expanding Eq. [A], we obtain

$$\mathbf{H}_a = \boldsymbol{\rho}_{ac} \times M \dot{\mathbf{r}}_c + \left(\sum m_i \boldsymbol{\rho}_{ci}\right) \times \dot{\mathbf{r}}_c + \boldsymbol{\rho}_{ac} \times \left(\sum m_i \dot{\boldsymbol{\rho}}_{ci}\right) + \sum \boldsymbol{\rho}_{ci}\, m_i \times \dot{\boldsymbol{\rho}}_{ci}\ .$$

The second and third terms on the right hand side of the above equation are zero as

$$\sum_i m_i \boldsymbol{\rho}_{ci} = 0\,,$$

as c is the centre of mass, and hence

$$\sum_i m_i \dot{\boldsymbol{\rho}}_{ci} = 0\,.$$

Therefore, we get

$$\mathbf{H}_a = \boldsymbol{\rho}_{ac} \times M\, \mathbf{V}_c + \mathbf{H}_c\,,$$

where $\mathbf{V}_c$ is the velocity of the centre of mass ($= d\mathbf{r}_c/dt$). The first term in the above represents the moment of momentum of the centre of mass (i.e. considering the system of particles to be replaced by a single particle of mass $M = \Sigma_i\, m_i$. The second term is the moment of momentum of the system of particles relative to the centre of mass c about the centre of mass and is given by

$$\mathbf{H}_c = \sum_i \boldsymbol{\rho}_{ci} \times m_i \dot{\boldsymbol{\rho}}_{ci}\ .$$

Now, let us start with Eq. [A] again and differentiate the moment of momentum vector $\mathbf{H}_a$ with respect to time to obtain

$$\begin{aligned}
\dot{\mathbf{H}}_a &= \sum_i (\dot{\boldsymbol{\rho}}_{ac} + \dot{\boldsymbol{\rho}}_{ci}) \times m_i (\dot{\mathbf{r}}_c + \dot{\boldsymbol{\rho}}_{ci}) + \sum_i \boldsymbol{\rho}_{ai} \times m_i \ddot{\mathbf{r}}_i \\
&= \sum_{i=1}^{n} \dot{\boldsymbol{\rho}}_{ac} \times m_i \dot{\mathbf{r}}_c + \sum_{i=1}^{n} \dot{\boldsymbol{\rho}}_{ac} \times m_i \dot{\boldsymbol{\rho}}_{ci} + \sum_{i=1}^{n} \dot{\boldsymbol{\rho}}_{ci} \times m_i \dot{\mathbf{r}}_c + \sum_{i=1}^{n} \dot{\boldsymbol{\rho}}_{ci} \times m_i \dot{\boldsymbol{\rho}}_{ci} + \sum_{i=1}^{n} \boldsymbol{\rho}_{ai} \times m_i \ddot{\mathbf{r}}_i \\
&= \dot{\boldsymbol{\rho}}_{ac} \times M \dot{\mathbf{r}}_c + \dot{\boldsymbol{\rho}}_{ac} \times \left(\sum_{i=1}^{n} m_i \dot{\boldsymbol{\rho}}_{ci}\right) + \left(\sum_{i=1}^{n} m_i \dot{\boldsymbol{\rho}}_{ci}\right) \times \dot{\mathbf{r}}_c + \sum_{i=1}^{n} \dot{\boldsymbol{\rho}}_{ci} \times m_i \dot{\boldsymbol{\rho}}_{ci} + \sum_{i=1}^{n} \boldsymbol{\rho}_{ai} \times m_i \ddot{\mathbf{r}}_i\ .
\end{aligned}$$

As c is the centre of mass, the second and third terms on the right hand side of the above equation vanish. The third term is zero as it involves the cross-product of a vector by itself. The last term can be simplified by considering the Newton's law for the i^{th} particles as (see Module 1)

$$\sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} F_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{f}_{ij} .$$

Taking moments of the above equation about point *a*, we obtain

$$\sum_{i=1}^{n} \boldsymbol{\rho}_{ai} \times m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} \boldsymbol{\rho}_{ai} \times \mathbf{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\rho}_{ai} \times \mathbf{f}_{ij} .$$

The last term in the above equation is zero as the moments of the pair of forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ cancel off. Hence, we have

$$\sum_{i=1}^{n} \boldsymbol{\rho}_{ai} \times m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} \boldsymbol{\rho}_{ai} \times \mathbf{F}_i = \mathbf{M}_a ,$$

where $\mathbf{M}_a$ is the total moment of all the external forces acting on the system of particles. Thus, we obtain the following important relationship

$$\boxed{\dot{\mathbf{H}}_a = \mathbf{M}_a + \dot{\boldsymbol{\rho}}_{ac} \times M \,\dot{\mathbf{r}}_c}.$$

Now, let us consider the following important cases.

CASE: 1 THE POINT A IS THE ORIGIN OF AN INERTIAL FRAME:

In this case, the point *a* happens to be the origin of the reference *O*. Therefore $\dot{\boldsymbol{\rho}}_{ac} = \dot{\mathbf{r}}_c$. Hence, we have for this case

$$\dot{\mathbf{H}}_O = \mathbf{M}_O .$$

CASE: 2 THE POINT A IS A FIXED POINT IN AN INERTIAL FRAME:

Here $\mathbf{r}_a + \boldsymbol{\rho}_{ac} = \mathbf{r}_c$ and $\dot{\mathbf{r}}_a = \mathbf{0}$. Hence $\dot{\boldsymbol{\rho}}_{ac} = \dot{\mathbf{r}}_c$ and we have

$$\dot{\mathbf{H}}_a = \mathbf{M}_a .$$

CASE 3: THE POINT A IS THE MOVING CENTRE OF MASS:

In this case, point *a* coincides with *c* and hence $\boldsymbol{\rho}_{ac} = \mathbf{0}$. Therefore, we have

$$\dot{\mathbf{H}}_c = \mathbf{M}_c .$$

CASE 4: IF "*A*" IS ACCELERATING EITHER TOWARDS OR AWAY FROM THE MASS CENTRE:

In this case too, it can be shown (with a little bit more hard-work) that

$$\dot{\mathbf{H}}_a = \mathbf{M}_a .$$

Ex: 2.33 A heavy chain of length 6 m lies on a light plate *A* which is freely rotating at an angular speed of 1rad/s. A channel *C* acts as a guide for the chain on the plate, and a stationary pipe acts as a guide for the chain below the plate. What is the speed of the chain after it moves 1.5 m starting from rest relative to the platform? Neglect friction and the angular momentums of the platform and the vertical section of the chain about its own axis. The weight of chain is 20 N/m.

A. TO GET THE ANGULAR SPEED AT THE SECOND INSTANT:

As $M_z = \dot{H}_z$ and $M_z = 0$, we have $(H_z)_1 = (H_z)_2$ (i.e the angular momentum is conserved). Thus, equating the angular momentum at the two instances, we obtain

$$\int_0^3 r V_{\theta_1} \frac{w}{g} dr = \int_0^{1.5} r V_{\theta_2} \frac{w}{g} dr .$$

As $V_{\theta_1} = r\omega_1$ and $V_{\theta_2} = r\omega_2$, we get

$$\omega_1 \int_0^3 r^2 dr = \omega_2 \int_0^{1.5} r^2 dr.$$

Since $\omega_1 = 1$ rad/s, we get from the above $\omega_2 = 8$ rad/s.

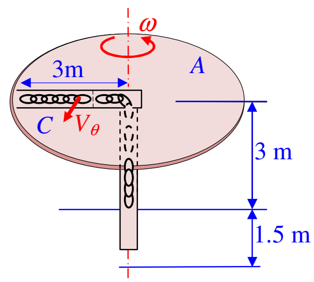

Figure 2.44

B. To find the speed of movement:

We may use the energy equation $\Delta PE + \Delta KE = 0$ as it is a conservative system. Thus, we obtain

$$[20\times1.5\times3 + 20\times4.5\times(2.25-1.5) - (20\times3\times3 + 20\times3\times1.5)] + 0.5\times20/9.81\times1.5\times V^2$$

$$+ 0.5\times20/9.81\times4.5\times V^2 + \frac{1}{2}\int_0^{1.5} \frac{20}{g}(r\omega_2)^2 dr - \left\{0 + \frac{1}{2}\int_0^{3} \frac{20}{g}(r\omega_1)^2 dr\right\} = 0,$$

which on simplification leads to $-11.25 + 6.116\ V^2 + 64.220 = 0$, from which we obtain the solution as $V = \underline{2.809}$ m/s.