Engineering Mechanics

Continued…

Mohammed Ameen, Ph.D

Professor of Civil Engineering

National Institute of Technology Calicut

Equivalent Force Systems

Equivalent vectors have the same capacity in a given situation

We shall, now, consider the equivalence requirements for force systems acting on a rigid body

The effect of forces on a rigid body is manifested in the motion (or lack of motion) of the body

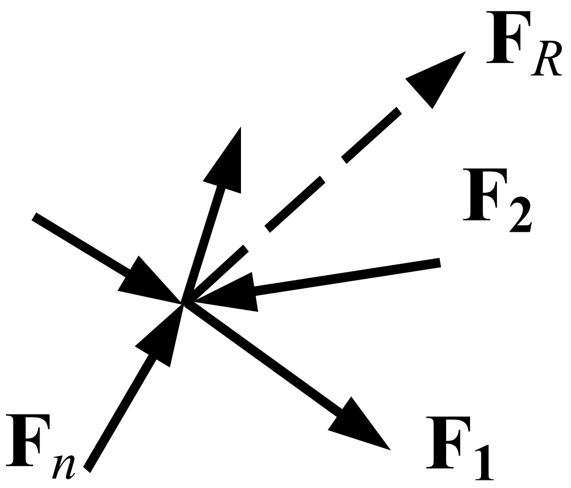
(i) each force system must exert an equal “push” or “pull” in any direction—i.e., the total force (sum of all forces) in each system is equal

(ii) Each force system must exert an equal “turning” action about any point in space—i.e., the moment of the total forces about any point is equal

Examples:

(i) A system of concurrent forces is equivalent to a single force which is equal to the sum of the forces

$$\mathbf{F}_R = \sum_{i=1}^{n} \mathbf{F}_i$$



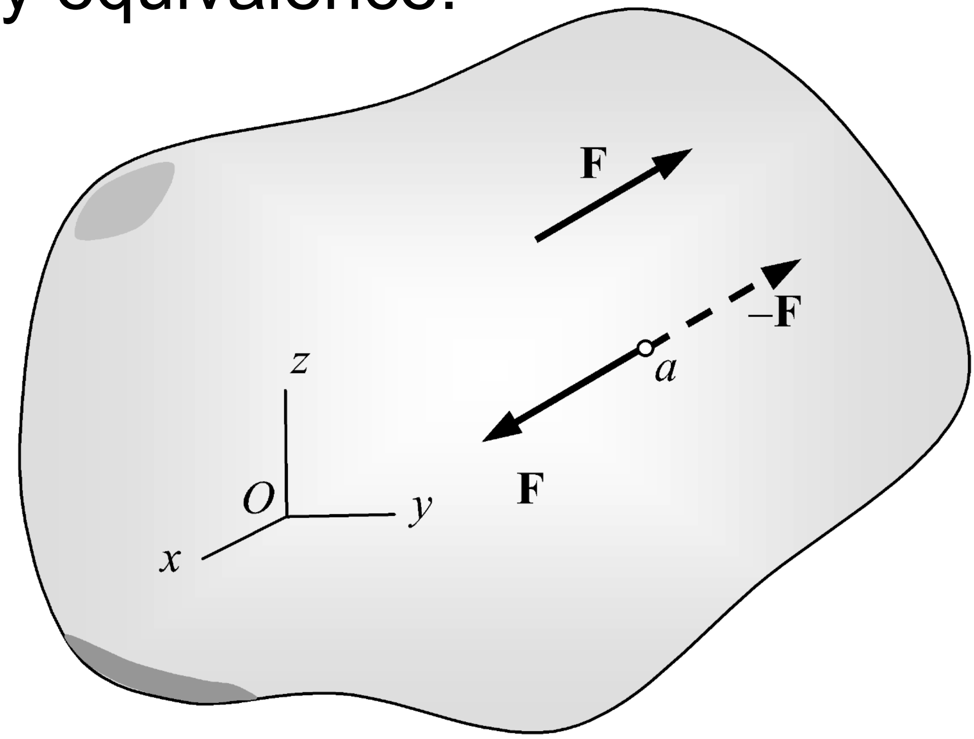
Thus, a system of concurrent forces acting on a rigid body can be replaced by its sum

(ii) A force may be moved along its line of action (i.e. *transmissible*)

(iii) The couples acting on a rigid body may be altered in any way without changing the couple moment

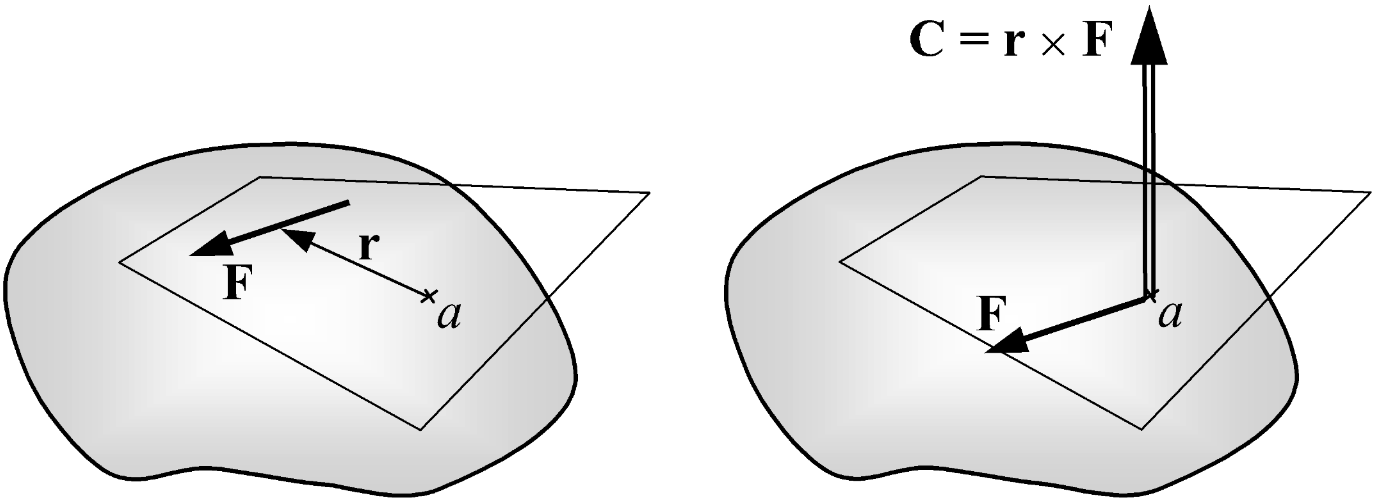
Translation of a Force to a Parallel Position

To move a force **F** acting on a rigid body to a parallel position while maintaining rigid body equivalence:

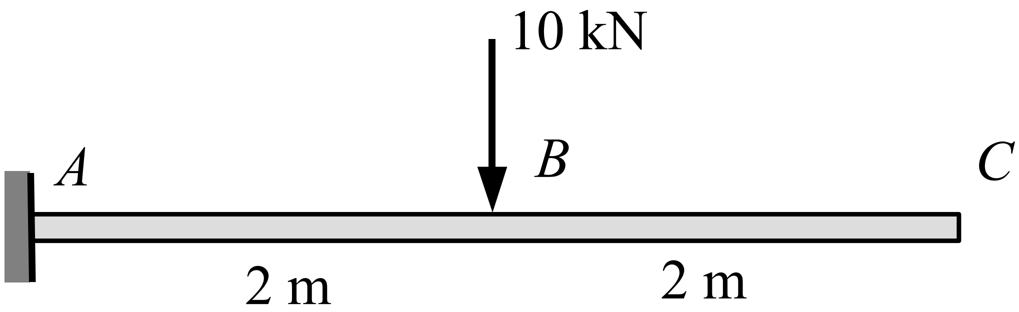


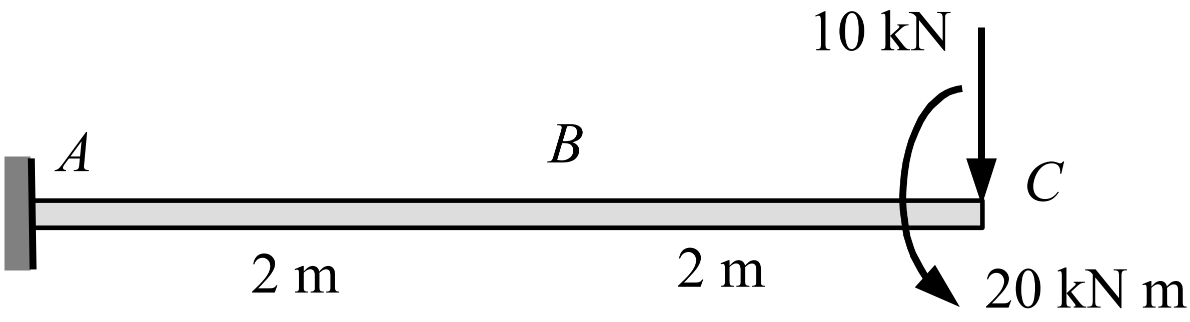
Add and subtract **F** at "a"

- Now, we are left with **F** through "a" and a couple of moment **M = r** × **F** (**r** is the position vector of old point w.r.t new point
- The couple moment could be moved to any parallel position (as it is a free vector)



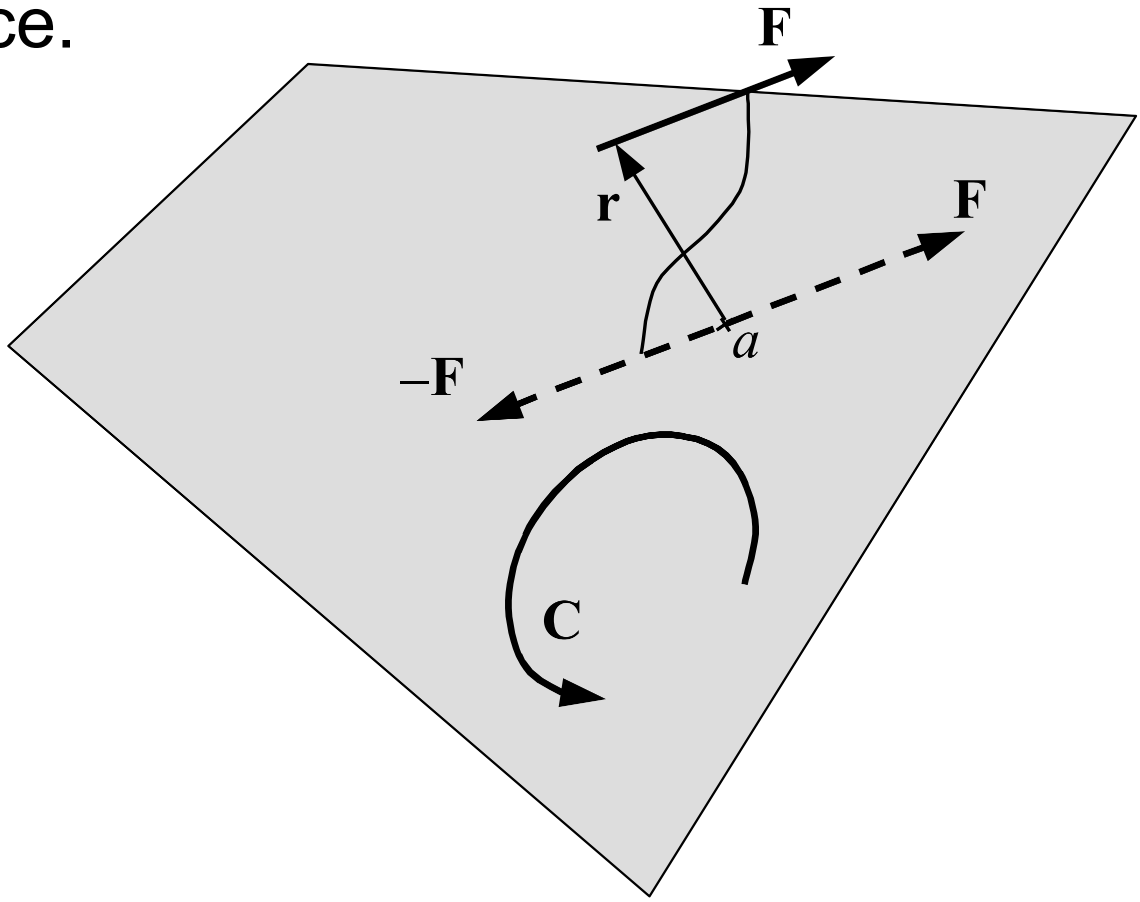
Example: Move the force to the tip and obtain an equivalent force system.



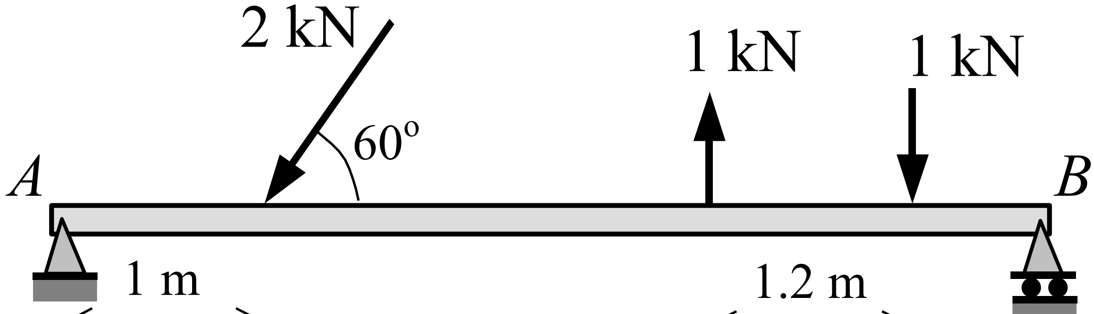


The reverse of the above procedure is also important.

A force and a couple in the same plane can be reduced to a single equivalent force.



Example: Move 2 kN to a new location so as to eliminate the couple.



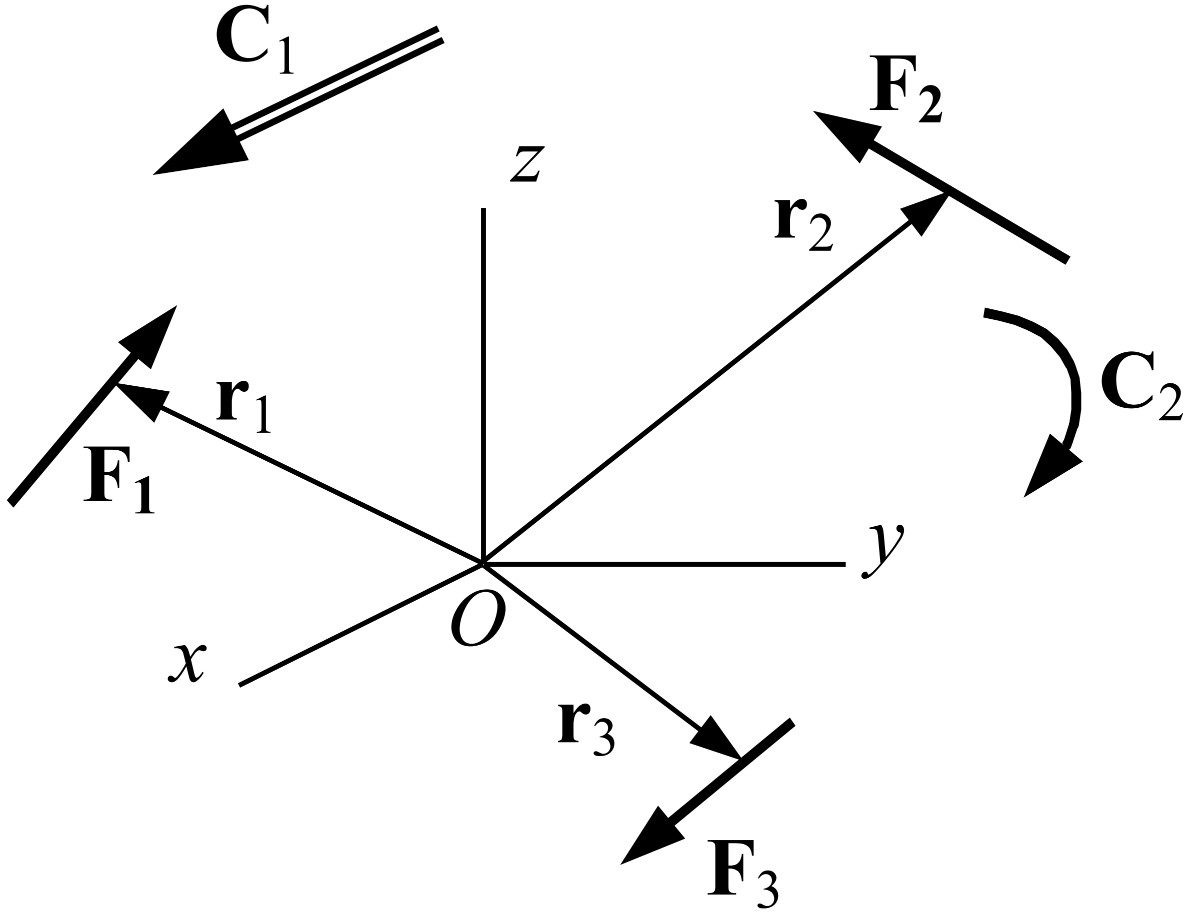
Resultant of a Force System

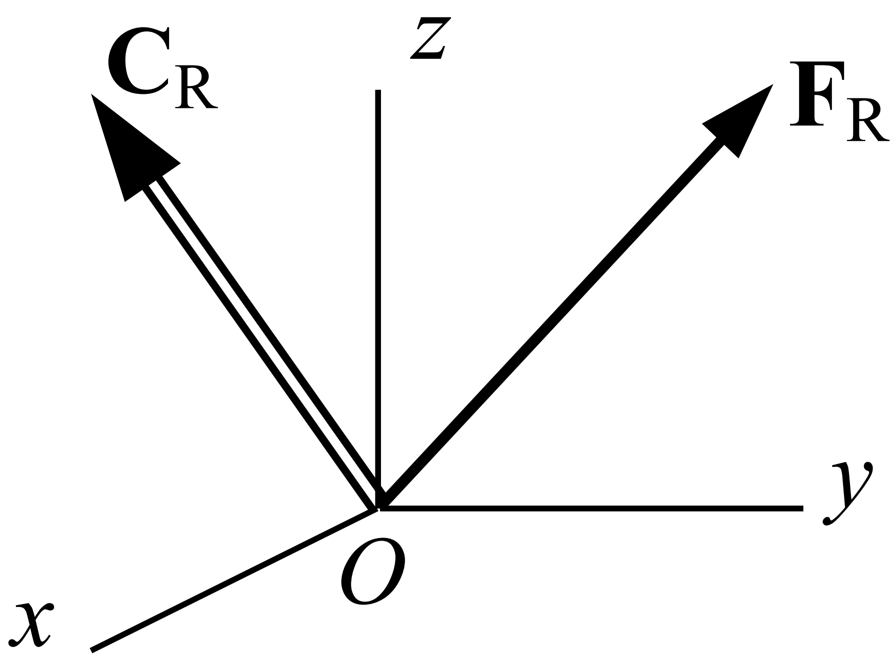
A resultant of a force system is a simpler equivalent force system.

In many problems, it is desirable to first establish a resultant.

General Case:

- Move all forces and couples (existing and the ones evolved due to the movement of the forces) to pass through any point
- We, thus, have a system of concurrent forces and a system of concurrent moments at the point
- This can be combined to obtain a single force $\mathbf{F_R}$ and a single couple $\mathbf{C_R}$ through the same point


C1
F2
z
r2
C2
r1
F1
y
O
x
r3
F3


CR
z
FR
y
O
x

Thus, any force system can be replaced by equivalents no more complex than a single force and a single couple moment.

In special cases, we may have simpler equivalents such as a single force or a single couple.

<u>Example</u>: Two forces and two couples are acting through the points as given below. Find the resultant of the system.

$\mathbf{F_1} = (3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ kN through (2, 5, 3) m

$\mathbf{F_2} = (10\mathbf{i} + 12\mathbf{k})$ kN thro' (1, 0, 3) m

$\mathbf{C_1} = (-\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$ kNm thro' (1, 6, 0) m

$\mathbf{C_2} = (6\mathbf{i} - \mathbf{j} + \mathbf{k})$ kNm thro' (2, 2, 1) m

$\mathbf{F_R} = \mathbf{F_1} + \mathbf{F_2} = 13\mathbf{i} + 5\mathbf{j} + 13\mathbf{k}$ kN

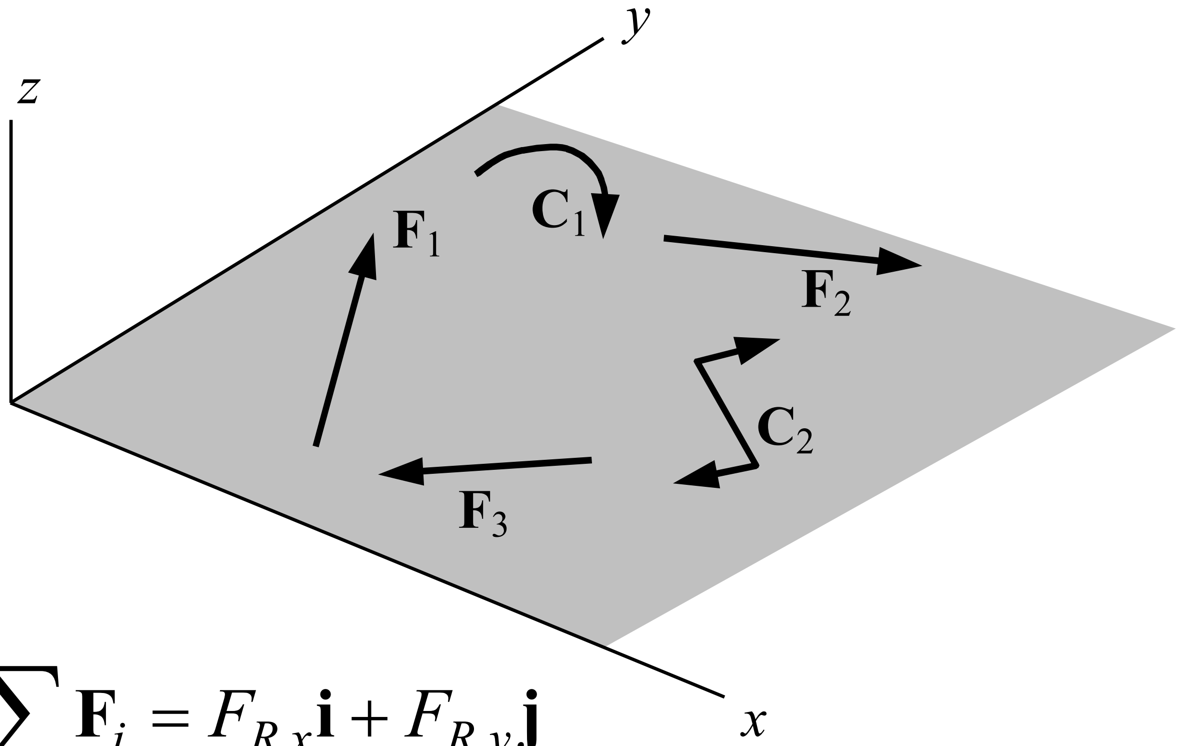
$\mathbf{C_R} = \mathbf{C_1} + \mathbf{C_1} + \mathbf{r_1} \times \mathbf{F_1} + \mathbf{r_2} \times \mathbf{F_2}$

$= (-5\mathbf{i} + 26\mathbf{j} + \mathbf{k})$ kNm

<u>Answer:</u> $\mathbf{F_R}$ and $\mathbf{C_R}$ acting thro' origin

Simple Resultants of Special Force Systems

A Coplanar Force System



$$\mathbf{F}_R = \sum_i \mathbf{F}_i = F_{R\,x}\mathbf{i} + F_{R\,y}\mathbf{j}$$

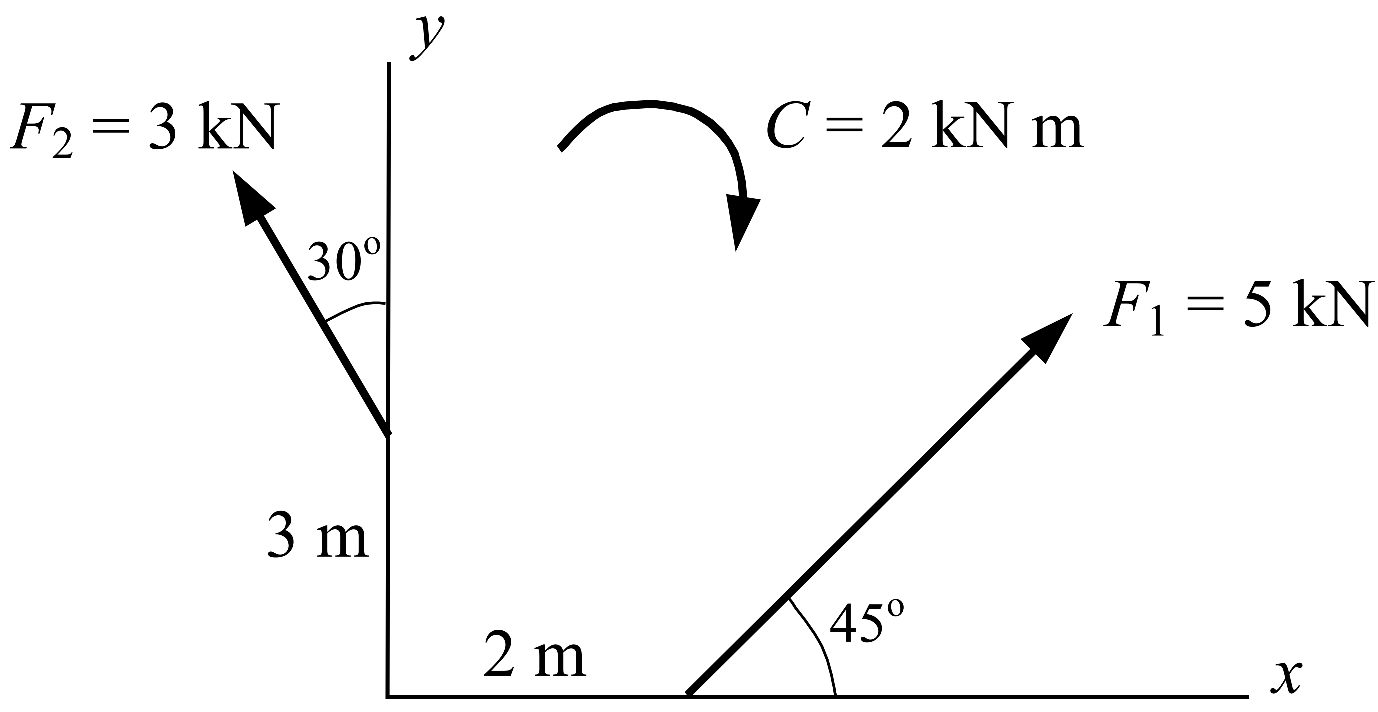
$$\mathbf{C}_R = \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_i \mathbf{C}_i = C_{R\,z}\mathbf{k}$$

If $\mathbf{F_R} \neq \mathbf{0}$, move it from origin to a new point so that $\mathbf{C_R}$ is eliminated;

If $\mathbf{F_R} = \mathbf{0}$, the resultant is either a single couple or the null vector.

The simplest resultant of a coplanar force system is a single force along a specified line of action, a single couple, or the null vector.

Example: Consider the system of forces shown acting on the *xy*-plane. Find the simplest resultant.



$\mathbf{F_1} = 5(\cos 45\ \mathbf{i} + \sin 45\ \mathbf{j})$ kN

$\mathbf{F_2} = (-3 \sin 45\ \mathbf{i} + 3 \cos 45\ \mathbf{j})$ kN

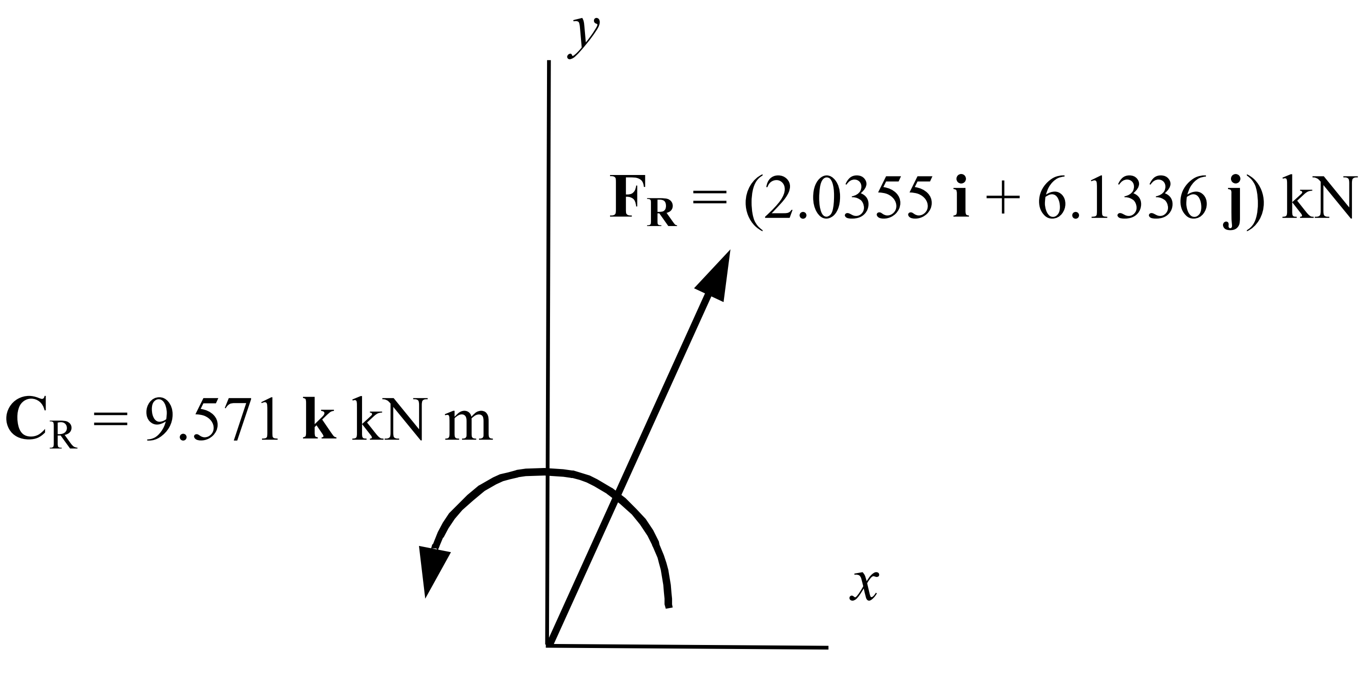
$\mathbf{C} = -2\ \mathbf{k}$ kN m

Move both the forces to the origin. Then,

$$\mathbf{F_R} = \mathbf{F_1} + \mathbf{F_2} = (2.0355\ \mathbf{i} + 6.1336\ \mathbf{j})\ \text{kN}$$

$$\mathbf{C}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 5\cos 45 & 5\sin 45 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ -3\sin 30 & 3\cos 30 & 0 \end{vmatrix} - 2\mathbf{k}$$

$$= 9.571\ \mathbf{k}\ \text{kN m}$$

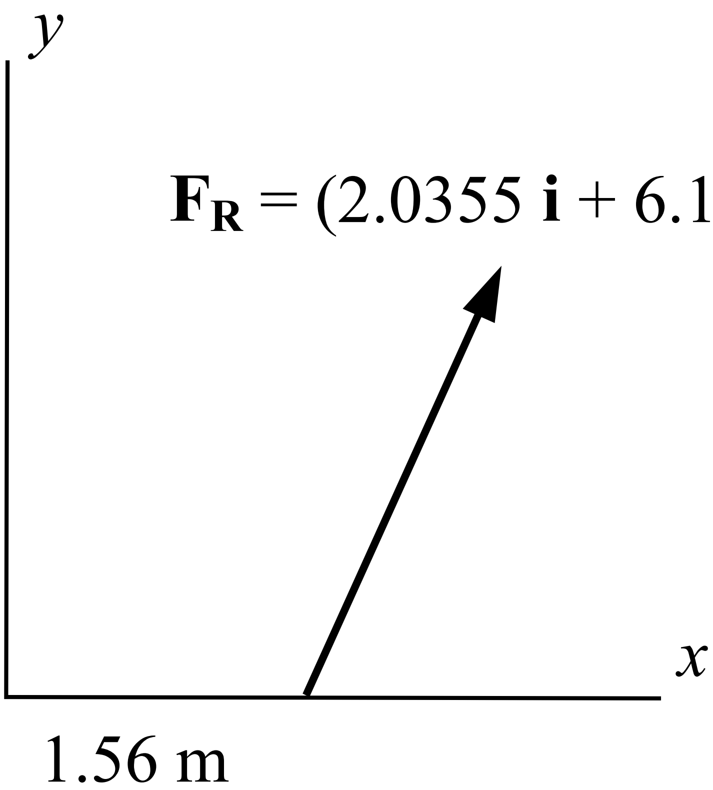


Move $\mathbf{F_R}$ to a new position, say $(x, 0)$ so that $\mathbf{C_R}$ gets eliminated. Thus, we have

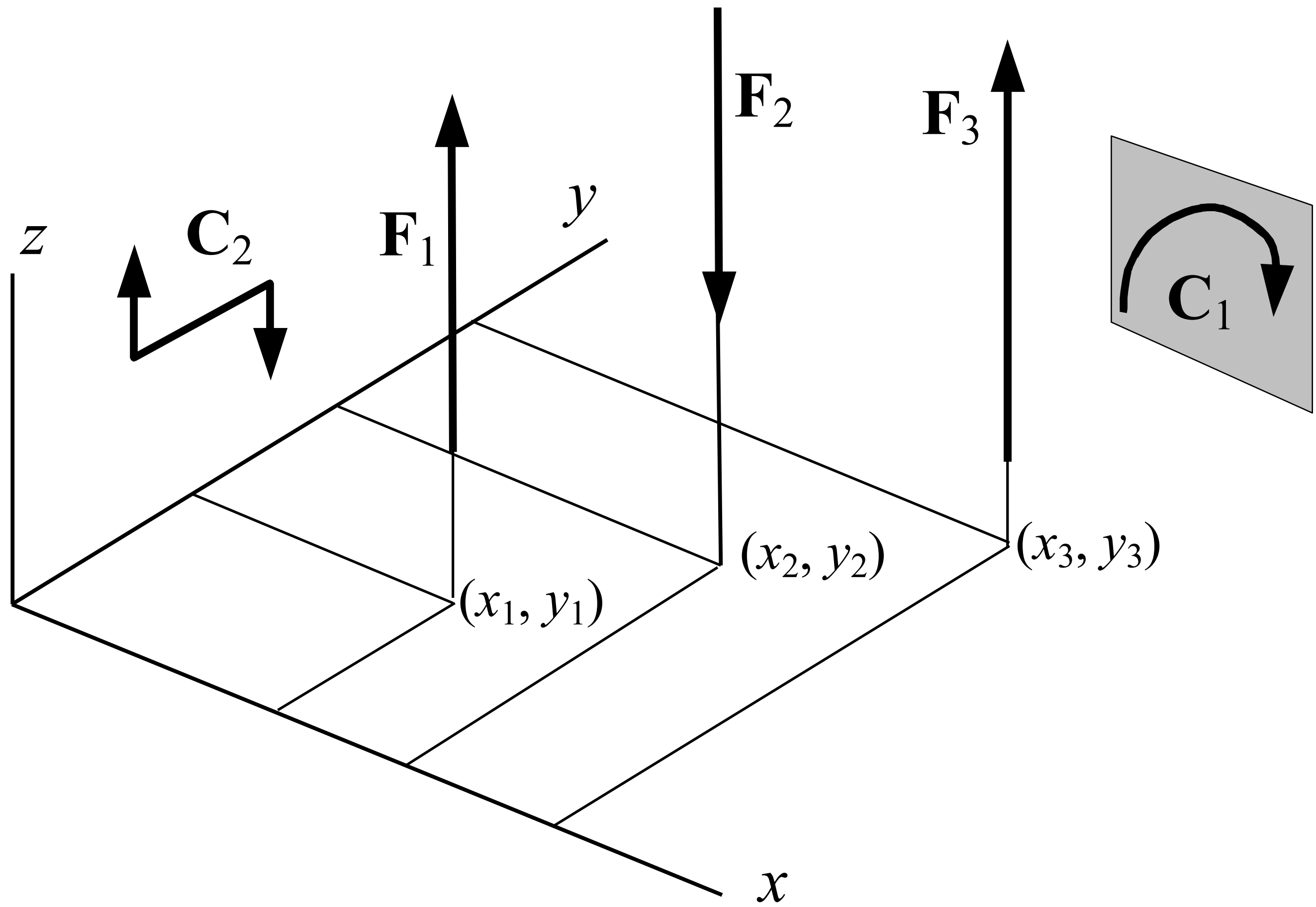
$$(-x\mathbf{i}) \times \mathbf{F}_R = -\mathbf{C}_R$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -x & 0 & 0 \\ 2.0355 & 6.1336 & 0 \end{vmatrix} = 9.571\mathbf{k}$$

Solving the above, we get $x = 1.56$ m



B Parallel Force Systems in Space

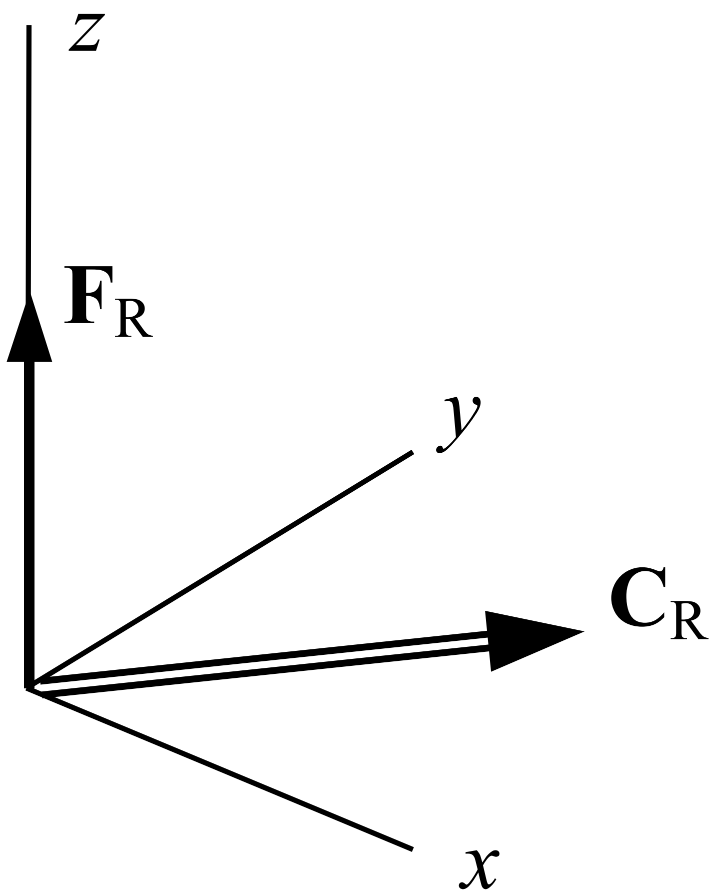


Move all forces to the origin to get:

$$\mathbf{F}_R = \left(\sum_{i=1}^{n} F_i \right) \mathbf{k}$$

$$\mathbf{C}_R = \sum_{j=1}^{m} \left(C_{jx}\mathbf{i} + C_{jy}\mathbf{j} \right) + \sum_{i=1}^{n} (x_i\mathbf{i} + y_i\mathbf{j}) \times F_i\mathbf{k}$$

$$= C_{Rx}\mathbf{i} + C_{Ry}\mathbf{j}$$

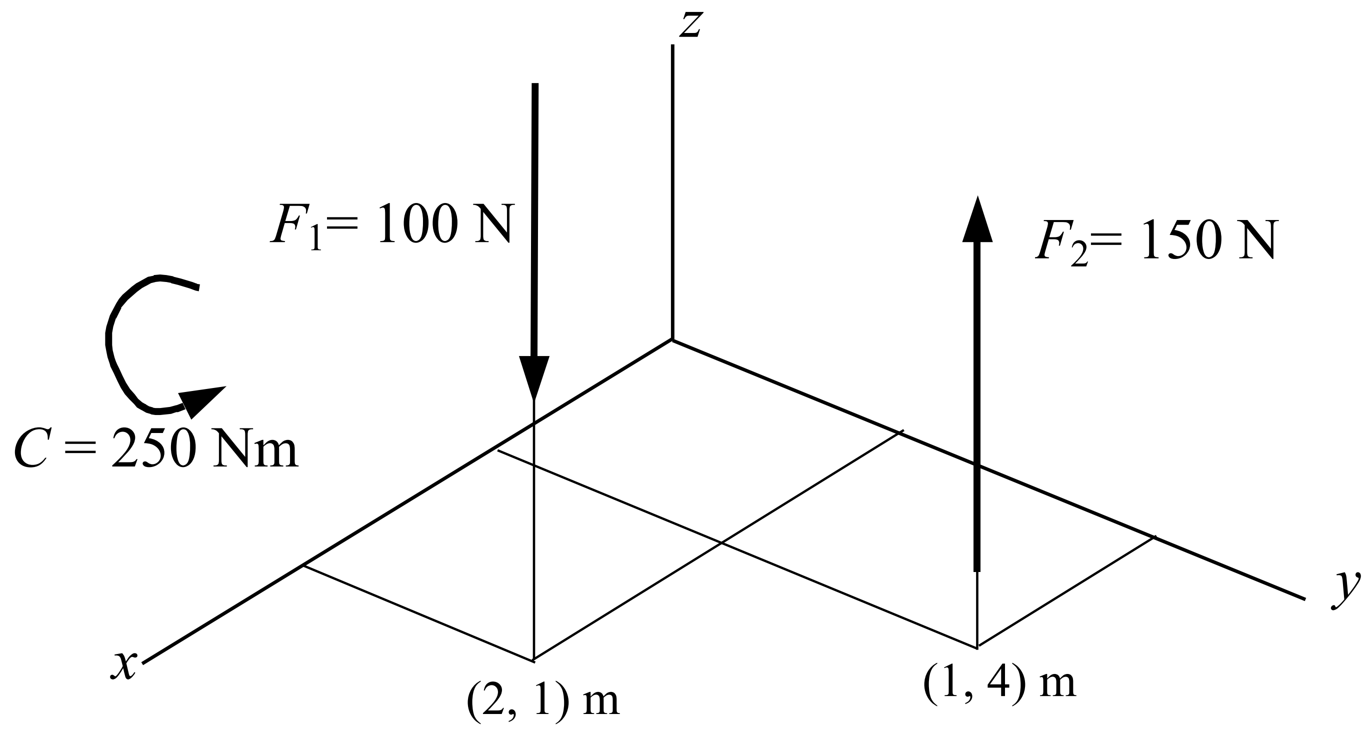


If $\mathbf{F_R} \neq \mathbf{0}$, we have $\mathbf{C_R}$ lying parallel to *xy* plane; we can eliminate $\mathbf{C_R}$ by moving $\mathbf{F_R}$ to a new location ($\bar{x}$, $\bar{y}$)

If $\mathbf{F_R} = \mathbf{0}$, the resultant is either a single couple or the null vector.

The simplest resultant of a parallel force system is a single force through a specified point, a single couple, or the null vector.

Example: Find the simplest resultant of the system of forces shown in figure.



$$\mathbf{F_R} = \mathbf{F_1} + \mathbf{F_2} = 50\,\mathbf{k}\text{ N}$$

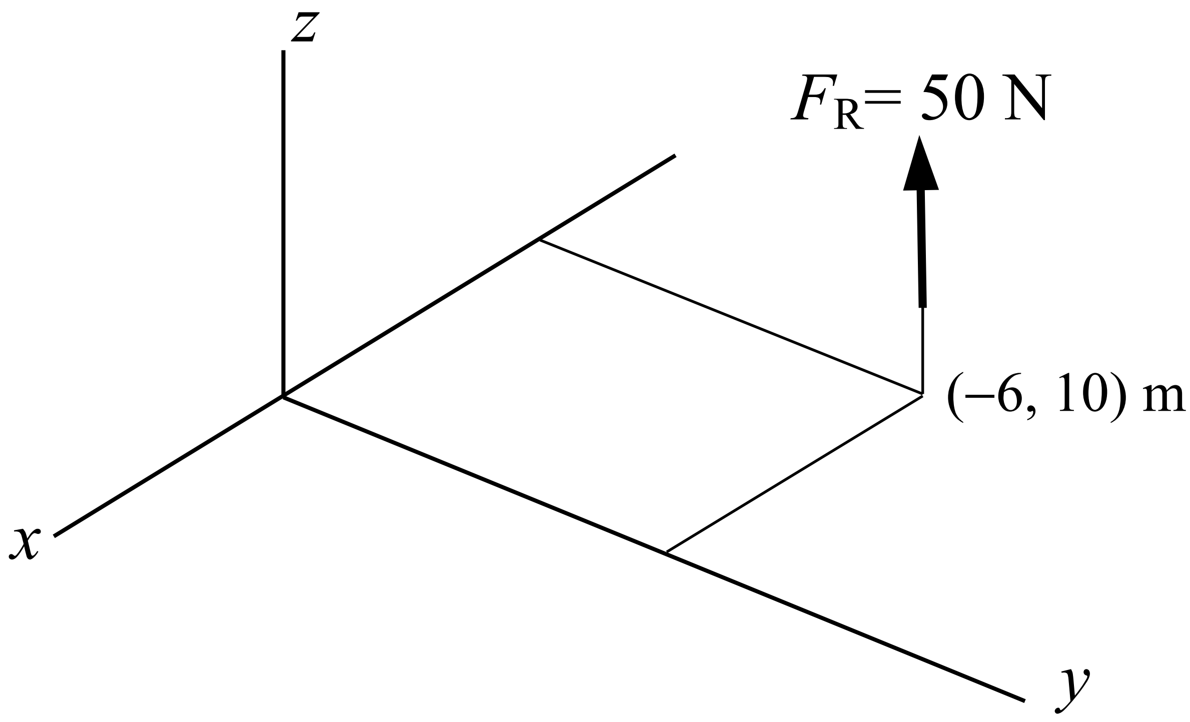
$$\mathbf{C}_R = 250\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 0 & 0 & -100 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 0 & 150 \end{vmatrix}$$

$$= (500\,\mathbf{i} + 300\,\mathbf{j})\text{ Nm}$$

Move $\mathbf{F_R}$ to a new location ($\bar{x}$, $\bar{y}$) so that $\mathbf{C_R}$ gets eliminated. Thus, we have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\bar{x} & -\bar{y} & 0 \\ 0 & 0 & 50 \end{vmatrix} = -\mathbf{C}_R = -500\mathbf{i} - 300\mathbf{j}$$

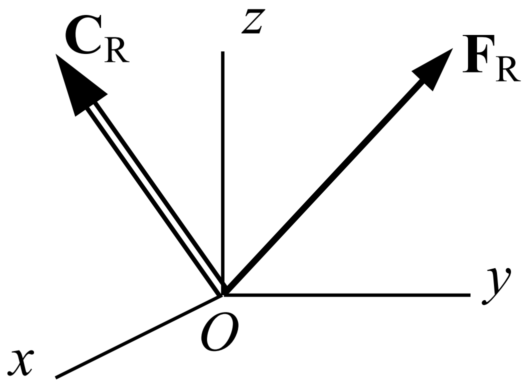
which gives $\bar{x} = -6$ m, $\bar{y} = 10$ m. Thus, the simplest resultant is 50 N force acting through (–6, 10) m along the *z*-axis.

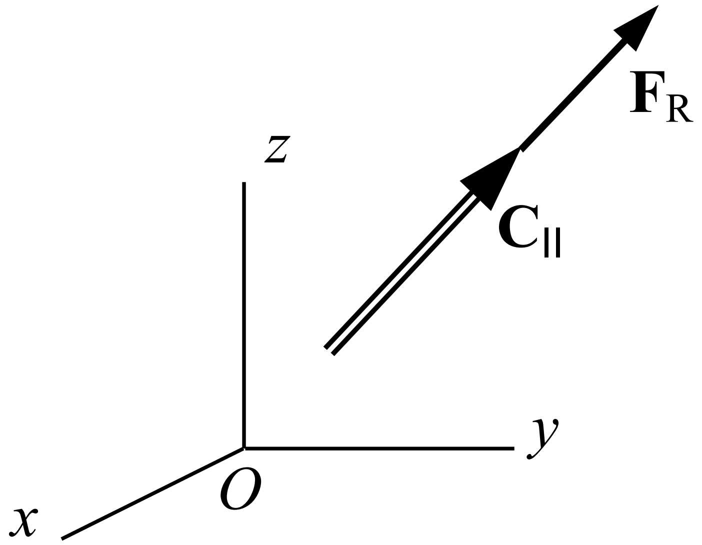


Check: Is the moment of $\mathbf{F_R}$ about *O* equal to moment of given force system about *O*?

General Force System—Wrench

In the case of a general force system, the simplest resultant consists of a single force $\mathbf{F_R}$ and a single couple $\mathbf{C_R}$ through the origin *O*.





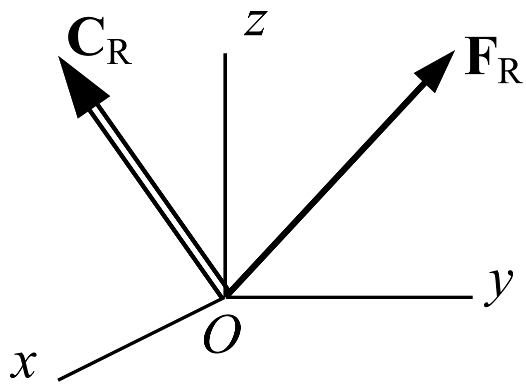
We can resolve $\mathbf{C_R}$ into two components $\mathbf{C_{II}}$ and $\mathbf{C_{\perp}}$, collinear and perpendicular to the force $\mathbf{F_R}$.

By moving $\mathbf{F_R}$ to a parallel position, we can eliminate $\mathbf{C_{\perp}}$.

Then, we are left with $\mathbf{F_R}$ and $\mathbf{C_{II}}$.

This is the <u>wrench</u>. It is the simplest resultant of any general force system.

Example: Reduce the following to a wrench:



$\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ N

$\mathbf{C} = -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ Nm

$\mathbf{f} = \mathbf{F}/|\mathbf{F}|$

$|\mathbf{C}_{II}| = \mathbf{C}.\mathbf{f} = 3.677$ Nm

$$\mathbf{C}_{II} = |\mathbf{C}_{II}|.\mathbf{f} = \frac{3.677}{\sqrt{50}}(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$$

$\mathbf{C}_{\perp} = \mathbf{C} - \mathbf{C}_{II} = -3.56\mathbf{i} + 0.92\mathbf{j} + 1.4\mathbf{k}$ Nm

Move **F** to a new position $(\bar{x}, \bar{y}, 0)$ such that $\mathbf{C}_{\perp}$ gets eliminated. Thus,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\bar{x} & -\bar{y} & 0 \\ 3 & 4 & 5 \end{vmatrix} = -\mathbf{C}_{\perp}$$

we get $\bar{x} = -0.184$ m, $\bar{y} = -0.712$ m.

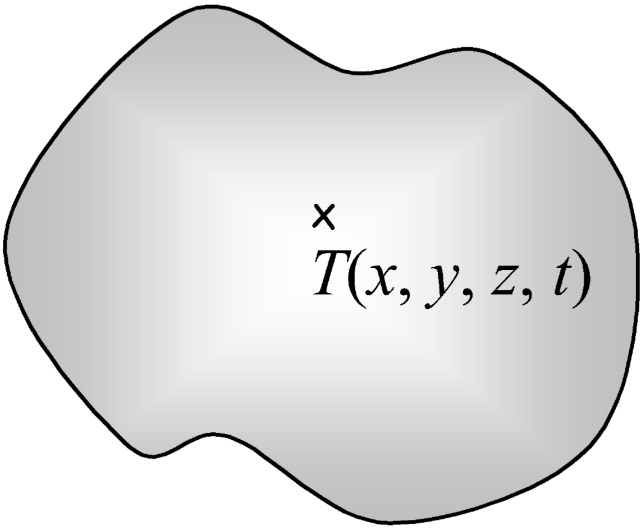
We are left with **F** and $\mathbf{C}_{II}$ through this point.

Distributed Force Systems

So far, we dealt with forces applied at discrete points—"Point Forces"

Scalars and vectors may be continuously distributed throughout a finite volume; these are called scalar and vector fields

Example: The temperature distribution in a body is a scalar field $T(x, y, z, t)$



Thus, once x, y, z and t are known, we can know the temperature if $T(x, y, z, t)$ is known as a function of x, y, z, t

A vector field can be represented as **F**(x, y, z, t). It can be written as three scalar fields $F_x(x, y, z, t)$, $F_y(x, y, z, t)$ and $F_z(x, y, z, t)$.

Thus, $\mathbf{F}(x,y,z,t) = F_x(x\ldots)\mathbf{i} + F_y\mathbf{j} + F_z\ \mathbf{k}$

Other examples of vector fields are velocity fields, displacement fields, etc.

<u>Body force distribution</u>: Forces that act per unit volume of the body.

Examples: gravity, force due to magnetic field, etc.

These forces are <u>not</u> due to contact of the body with any other body.

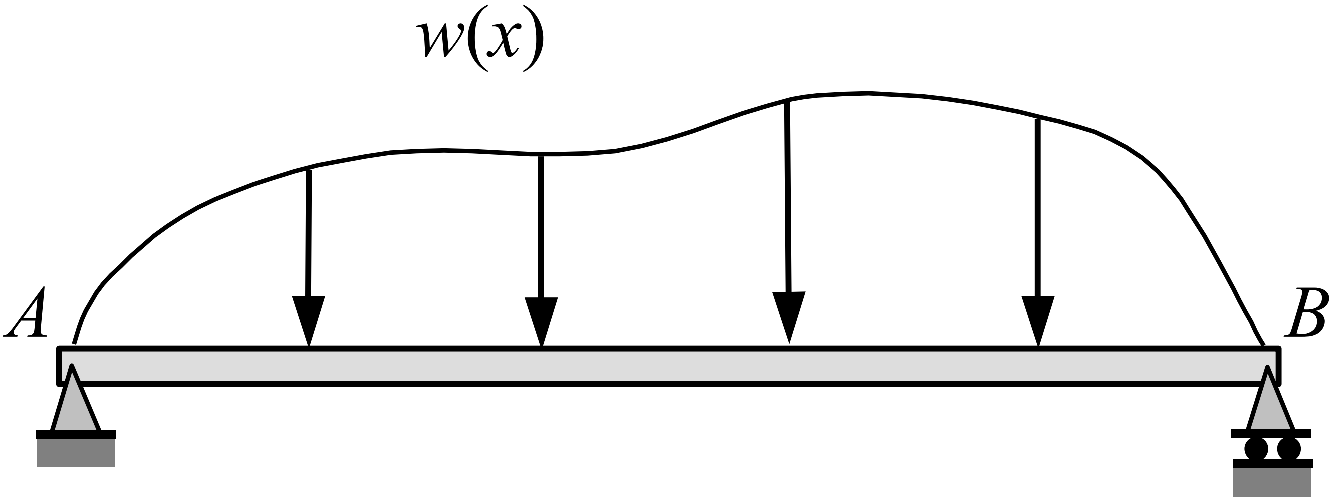
<u>Surface force distribution</u>: Forces that act per unit area of the surface of a body.

These are caused due to contact.

Example: Fluid pressure acting on an immersed body.

Line loads: Forces distributed over a line.

Example: Continuous load on a beam.



$w(x)$ is the intensity of load in N/m.

The force on an element is $w(x)\,dx$.

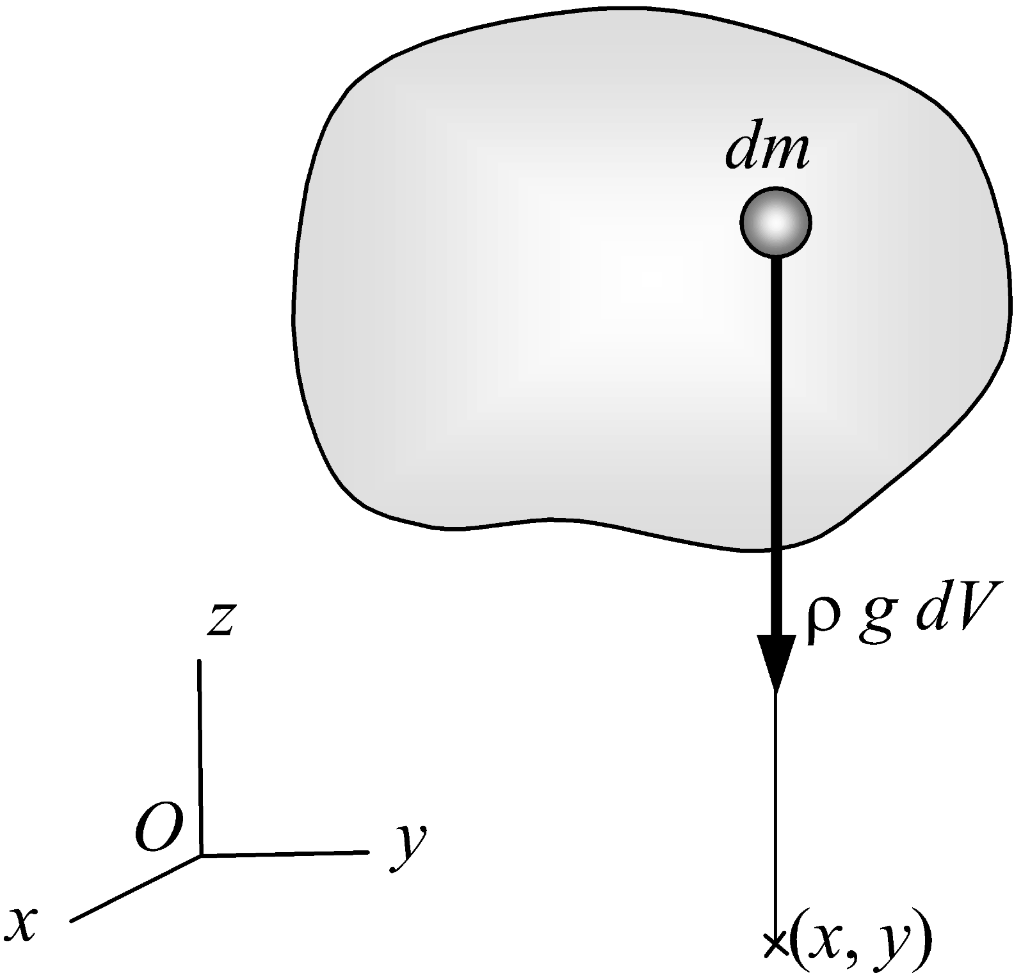
Each of the above distributed force system can be considered as an infinite number of point forces.

Hence, all the results regarding the resultants of special force systems are valid in these cases too.

A Parallel Body Force System—Centre of Gravity

Consider a rigid body whose mass density is given as $\rho(x, y, z)$.

It is acted on by gravity; and thus, results in a distributed force field.



Since this is a parallel force field in space, the resultant will be a single force $\mathbf{F_R}$ with a prescribed line of action.

On a differential element of volume *dV*, the infinitesimal force is $-g\,\rho\,dV\,\mathbf{k}$.

Hence, the resultant force is given by

$$\mathbf{F}_R = -\int_V g\,\rho\,dV\,\mathbf{k} = -g\mathbf{k}\int_V \rho\,dV = -gM\,\mathbf{k}$$

where *M* is the total mass of the body.

Let $\mathbf{F_R}$ pass through $(\bar{x}, \bar{y})$ on the xy-plane.

Taking moments of forces about the *x*-axis, we get

$$-F_R\bar{y} = -g\int_V y\rho\,dV$$

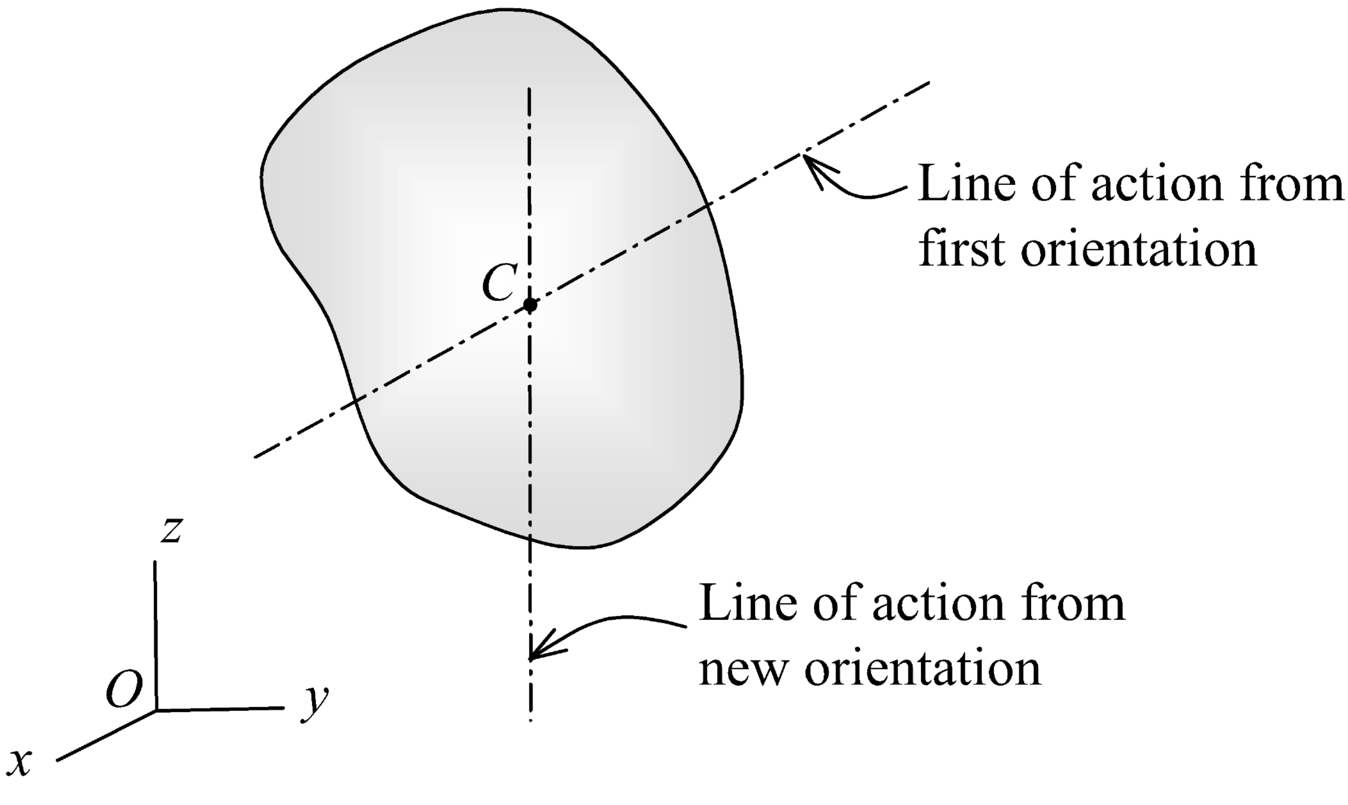
$$-F_R\bar{x} = -g\int_V x\rho\,dV$$

$$\bar{x} = \frac{\int_V x\rho\,dV}{M}; \qquad \bar{y} = \frac{\int_V y\rho\,dV}{M}$$

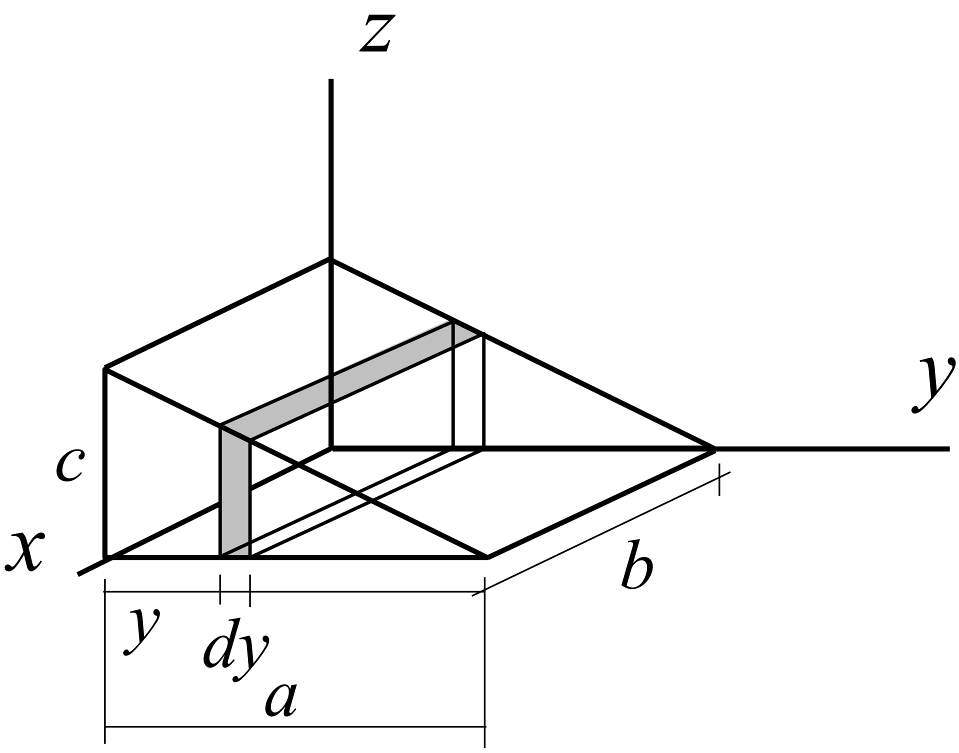
Reorient the body

Find the new line of action of $\mathbf{F_R}$

The intersection of the two lines of action provides the "centre of gravity"



<u>Example</u>: Find the centre of gravity of a triangular prism with sides a, b and c.



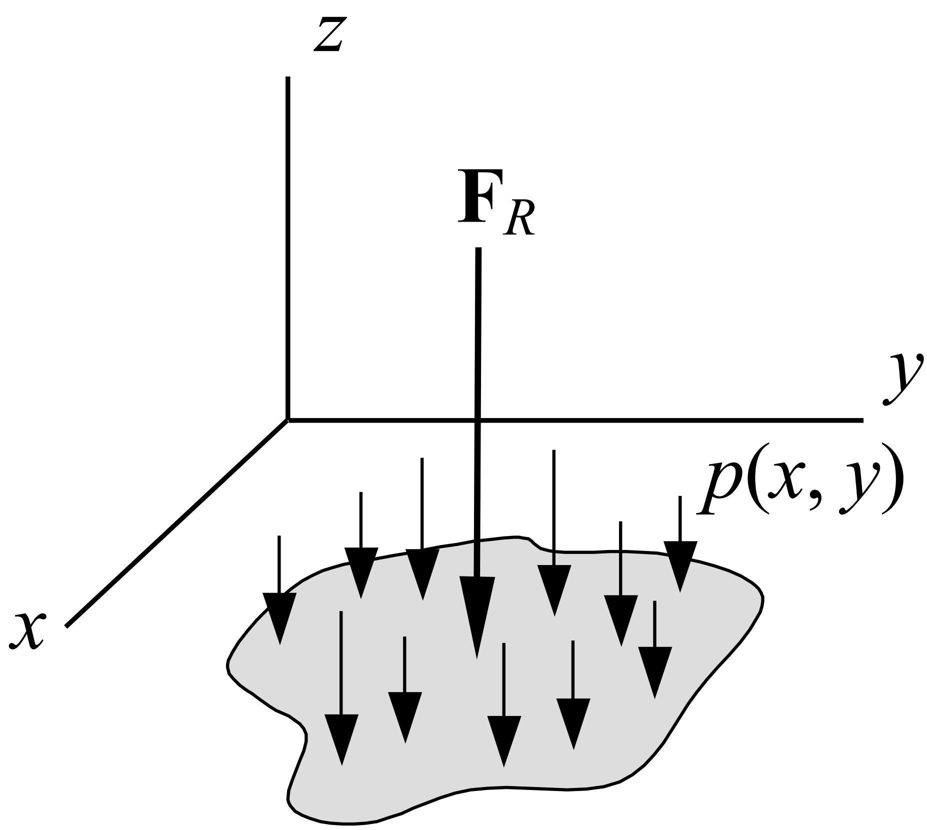
$\mathbf{F_R}$ = weight = $0.5 \times a\, b\, c\, g\, \rho\, \mathbf{k}$.

$$F_R \bar{y} = \int_0^a y \rho\, g\, dV = \int_0^a y \rho\, g\, \{dy \frac{c}{a}(a-y)b\}$$

$$= \frac{a^2 bc\rho\, g}{6}$$

Hence, $\bar{y} = \dfrac{a}{3}$

B Parallel Force Distribution over a Plane

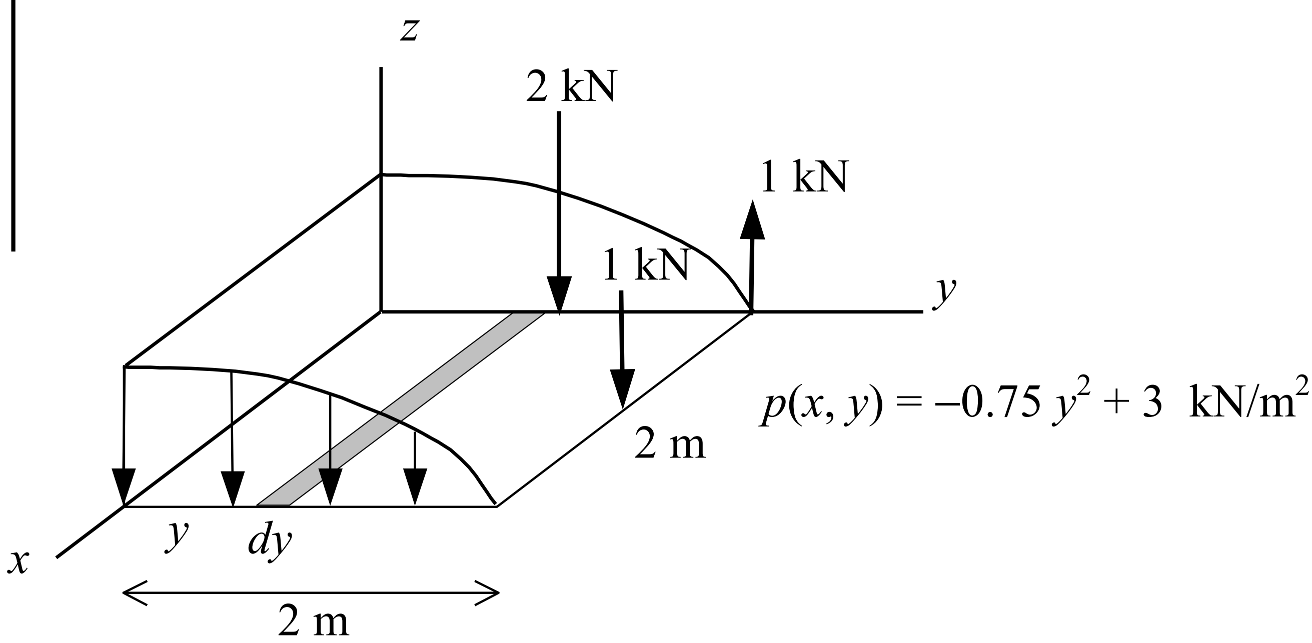


$$\mathbf{F}_R = -\int_A p(x,y)\, dA\ \mathbf{k} = -\int_A p(x,y)\, dx\, dy\ \mathbf{k}$$

$(\bar{x}, \bar{y})$ is the centre of pressure:

$$\bar{x} = \frac{\int_A p\, x\, dA}{\int_A p\, dA} \qquad \bar{y} = \frac{\int_A p\, y\, dA}{\int_A p\, dA}$$

Example: Determine the simplest resultant of the following



$$\mathbf{F}_R = -2\mathbf{k} - \int_0^2 2\,dy\,(-0.75\,y^2 + 3)\,\mathbf{k} = -10\,\mathbf{k}\ \text{kN}$$

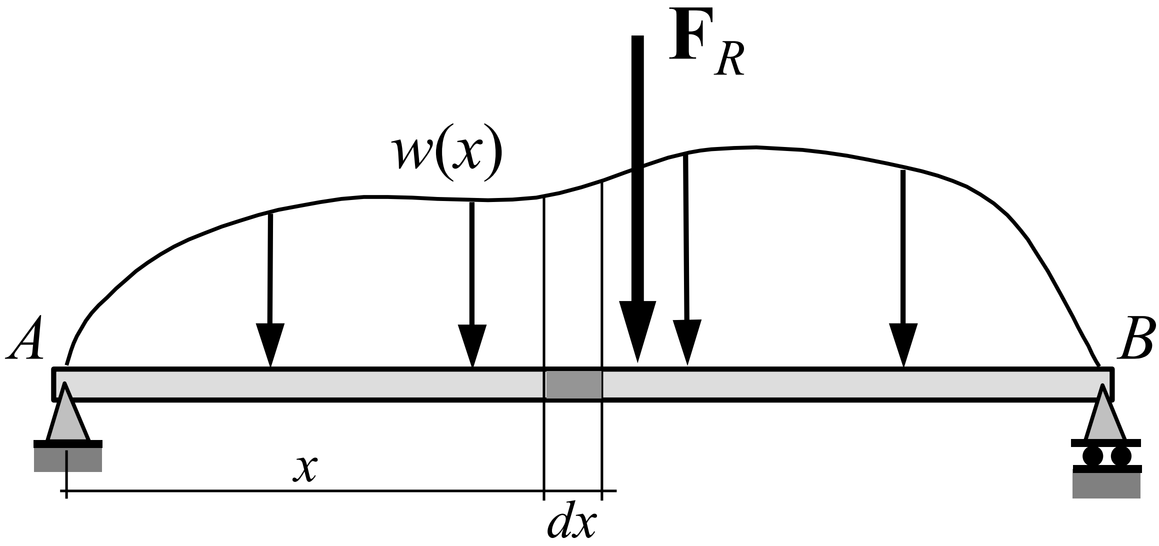
$$10\,\bar{x} = 1 + \int_0^2 2\,dy\,(-0.75\,y^2 + 3)\,1 = 9\ \text{kNm}$$

$$10\,\bar{y} = 2 \times 1 + \int_0^2 2\,dy\,(-0.75\,y^2 + 3)\,y = 8\ \text{kNm}$$

Hence, we have

$\bar{x} = 0.9$ m and $\bar{y} = 0.8$ m

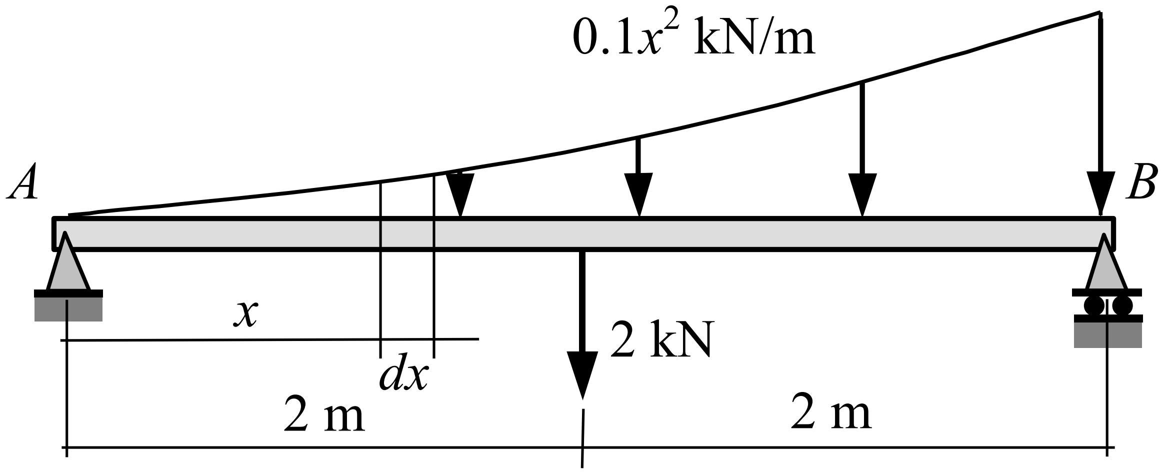
C Coplanar Parallel Force System—Line Loads



$$F_R = \int_0^l w(x)\, dx$$

$$\bar{x} = \frac{\int_0^l w(x)\, x\, dx}{F_R}$$

Example: Determine the simplest resultant of the load acting on the beam



$$F_R = 2 + \int_0^4 0.1\,x^2\,dx = 4.133\text{ kN}$$

$$4.133\,\bar{x} = 2\times 2 + \int_0^4 0.1\,x^2 x\,dx = 10.4\text{ kNm}$$

$$\bar{x} = 2.516\text{ m}$$